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Curvature formula for the space of 2-d conformal field theories

Daniel Friedan[a,b] **and Anatoly Konechny**[c,d]

[a] *Department of Physics and Astronomy, Rutgers, The State University of New Jersey, Piscataway, New Jersey 08854-8019 U.S.A.*

[b] *Natural Science Institute, The University of Iceland, Reykjavik, Iceland*

[c] *Department of Mathematics, Heriot-Watt University, Riccarton, Edinburgh, EH14 4AS, U.K.*

[d] *Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.*

E-mail: friedan@physics.rutgers.edu, anatolyk@ma.hw.ac.uk

Abstract: We derive a formula for the curvature tensor of the natural Riemannian metric on the space of two-dimensional conformal field theories and also a formula for the curvature tensor of the space of boundary conformal field theories.

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1 Introduction

Following up on the work of Kutasov [1], we derive a formula

$$R_{ijkl} = \mathrm{RV}\int \frac{d^2\eta}{2\pi}\,(-\ln|\eta|)\,\langle\,\phi_i(1)\phi_j(\eta)\phi_k(\infty)\phi_l(0)\,\rangle_c \tag{1.1}$$

for the curvature tensor of the Zamolodchikov metric on the space of two dimensional conformal field theories (CFT's), and a similar formula

$$R_{abcd} = \mathrm{RV} \int_{-\infty}^{\infty} d\eta \, (-\ln|\eta|) \, \left[\langle \psi_a(1)\psi_b(\eta)\psi_c(\infty)\psi_d(0) \rangle_c \right. \tag{1.2}$$

$$\left. + \langle \psi_a(0)\psi_b(1-\eta)\psi_c(\infty)\psi_d(1) \rangle_c \right] \tag{1.3}$$

for the space of boundary conformal field theories.

In the first formula, the ϕ_i are exactly marginal fields in the conformal field theory at which the curvature tensor is calculated. In the second formula, the ψ_a are exactly marginal boundary fields of the boundary conformal field theory where the curvature tensor is calculated. In both formulas, $\langle \cdots \rangle_c$ is the connected four-point correlation function. The integrands have singularities at $\eta = 0, 1, \infty$. The letters 'RV' denote a particular prescription for regularizing and subtracting the divergences — a hard-sphere (point-splitting) cutoff followed by minimal subtraction of the divergences. The formulas are derived under mild technical assumptions explained in section 2 below. The main limitation is the exclusion of redundant (total derivative) fields. Generically there is no reason to consider redundant fields. However, as we explain in appendix A, redundant fields cannot be avoided in a a neighborhood of a CFT with continuous symmetry. In string theory this phenomenon is known as the string Higgs effect. Appendix A explains the underlying two-dimensional physics.

We study the local geometry abstractly, in terms of the correlation functions of the conformal field theory at which we are calculating the curvature. Our work is motivated by a desire to get better control over the geometry of spaces of conformal field theories and of string theory vacua. The N=2 superconformal theories related to Calabi-Yau manifolds provide well-studied examples of spaces of CFT's. These 2-d conformal field theories provide string compactifications. The geometry of their moduli spaces has been determined from consideration of the low-energy effective field theory corresponding to the low-energy string scattering amplitudes. For these models, a formula expressing the curvature in terms of the N=2 CFT data was derived from the low energy effective action [2] (see formulas (3.37) in that paper). More recently in [3] the curvature was computed explicitly for a number of examples with N=2 and N=4 supersymmetry. Our formula (1.1) is general, not restricted to N=2 CFT's. We derive the general curvature formulas (1.1), (1.3) directly from 2-d conformal field theory in order to avoid assuming the low energy effective action. We are interested in a general derivation directly from 2-d CFT partly because string theory requires restrictions on the values of the conformal central charge c, but mainly because there is no complete proof of the correspondence between the low energy effective action and the string amplitudes. Our calculations can be considered as providing a point of support for that correspondence.

We check the curvature formulas in some of the few known families of conformal field theories where the curvature can be computed directly.

Formulas (1.1) and (1.3) can be derived in a variety of ways. We derive the bulk curvature formula from the 2-d conformal anomaly using a slightly novel analytic regularization scheme for conformal perturbation theory. We derive the boundary curvature formula by

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directly computing second derivatives of the metric using a sharp point-splitting cutoff. We chose such different methods hoping that the techniques might be useful elsewhere. We put particular emphasis on carefully deriving the particular regularization and subtraction prescription for the integrals in the curvature formulas.

A speculative motivation for deriving the curvature formula is the possibility that it could be used to prove the claim made in [4] that the natural metric on the space of supersymmetric string vacua satisfies an Einstein equation $R_{ij} = \frac{1}{4} g_{ij}$.

2 The space of conformal field theories

In this paper, a 2-d conformal field theory is a unitary euclidean quantum field theory on the complex plane. The trace of the stress-energy tensor vanishes, implying locally conserved conformal currents. The space of conformal field theories is — modulo some technical assumptions — the set of fixed points of the renormalization group acting on the space of 2-d unitary quantum field theories. We are interested in the local geometry of the space of conformal field theories in the neighborhood of an arbitrary given CFT, the *reference CFT*. We suppose that the reference CFT has unbroken global conformal invariance[1] and a discrete spectrum of conformal dimensions.

A family of CFT's in a neighborhood of the reference CFT is described by coordinates given by coupling constants λ^i. The λ^i parametrize perturbations of the action of the reference field theory that preserve scale invariance. The partition function of the perturbed theory is

$$Z(\lambda) = Z(0) \langle\, e^{\frac{1}{2\pi} \int d^2z \lambda^i \phi_i(z)} \,\rangle \tag{2.1}$$

where $Z(0)$ is the partition function of the reference CFT. The $\phi_i(z)$ are local fields in the reference CFT, and $\langle \cdots \rangle$ is the expectation value in the reference CFT.

We make the following technical assumptions

Assumption 1 *The $\phi_i(z)$ are dimension 2 scalar fields in the reference CFT.*

Assumption 2 *The $\phi_i\,\phi_j$ operator product expansions (OPE's) contain no dimension 2 scalar fields.*

Assumption 3 *The $\phi_i\,\phi_j$ OPE's contain no dimension 1, spin 1 currents.*

Assumptions 1 and 2 imply that the beta functions for the couplings λ^i vanish at least through the second order. Although assumptions 2 and 3 restrict the $\phi_i\,\phi_j$ OPE's, we emphasize that there are no other restrictions. In particular, relevant scalar fields can appear in the OPE's. Scale invariance is preserved by minimally subtracting the associated power divergences. In more general terms we adjust the couplings for the relevant fields

[1] The supposition of global conformal invariance usually goes unspoken. It avoids the possibility of a locally conformal field theory which, on the 2-d plane, exhibits spontaneously broken conformal invariance. See [5, 6] for examples and further discussion. Global conformal invariance and unitarity on the plane together imply unitarity of the radial quantization. The self-adjointness of the dilation and rotation operators then implies that the local fields can be expanded in scaling fields of definite dimension and spin.

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so that their beta functions are zero. This is especially simple in the minimal subtraction scheme. The zeroes of the beta functions $\beta^{i'}$ for the relevant couplings $\lambda^{i'}$ are at $\lambda^{i'} = 0$. If we used instead a non-minimal scheme, the zeroes of $\beta^{i'}$ could be at non-vanishing values of $\lambda^{i'}$. For example in a scheme in which $\beta^{i'} = \delta_{i'}\lambda^{i'} + C^{i'}_{jk}\lambda^j\lambda^k$ we should set $\lambda^{i'} = -C^{i'}_{jk}\lambda^j\lambda^k/\delta_{i'}$ to preserve conformal invariance. The non-zero relevant couplings do not contribute to the beta function for the marginal couplings by the usual dimensional analysis argument — any such contributions would have negative dimension coefficients.

Assumption 1 excludes any perturbations by total derivative operators. Such a perturbation only amounts to a redefinition of the local fields and a corresponding reparametrization of the space of CFT's. None of the physical properties change — the perturbed CFT is equivalent to the unperturbed theory. These perturbations are called *redundant*. In Lagrangian quantum field theory, they arise from perturbations by terms that vanish by the equations of motion. Assumption 1 in conjunction with unitarity and global conformal invariance implies that the ϕ_i are primary fields and therefore cannot be total derivatives.

A dimension 1, spin 1 current is necessarily conserved. If any such current is present in the reference CFT, assumption 3 states that none of the fields ϕ_i are charged under the corresponding continuous symmetry. If there were such a charged perturbation, it would break the continuous symmetry. We show in appendix A that, at first order in the symmetry breaking perturbation, a certain linear combination of the ϕ_i becomes a total derivative. Thus assumptions 1 and 3 allow us to disregard systematically the possibility of redundant perturbations. One could relax our assumptions to allow for redundant perturbations at the cost of technical complication.

We are studying the curvature tensor on a smooth family of CFT's. The beta function will vanish identically on such a family, but we only need to assume that it vanishes through second order at the reference CFT. This is enough to describe the curvature tensor at a generic point of the moduli space of CFT's (the space of *all* equivalence classes of CFT's). At generic points the moduli space is smooth.

Singularities in the moduli space can take various forms. There are singular points where a number of smooth families of CFTs intersect. Our curvature formula applies to each of the intersecting families. There are singular points in the moduli space which are CFT's with discrete symmetries, under which the perturbations ϕ_i transform nontrivially. The discrete symmetries act as equivalence transformations on the smooth family of perturbed theories. The moduli space of CFT's is the quotient orbifold. We are calculating the curvature tensor on the smooth family before the discrete quotient is taken. Another class of singular points in the moduli space arises from CFT's with continuous symmetries where some of the perturbations are charged. Again, the symmetries act as equivalence transformations on the smooth family of perturbations. Handling this case would require including redundant operators.

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3 The metric and the curvature tensor

The natural riemannian metric $g_{ij}(\lambda)$ on the family of CFT's is extracted from the two-point correlation function in the perturbed CFT,

$$\langle\, \phi_i(z)\, \phi_j(w)\, \rangle_\lambda = g_{ij}(\lambda)\, |z-w|^{-4}\,. \tag{3.1}$$

Scale invariance dictates the form of the two-point function. The coefficient $g_{ij}(0)$ is the riemannian metric at the reference CFT.

To calculate the curvature tensor, we need the first and second derivatives of the metric at the reference CFT. These are calculated in the conformal perturbation series, which is the expansion of the partition function and the correlation functions in powers of the coupling constants λ^i. The conformal perturbation series is encoded in the generating functional

$$Z(\lambda) = Z(0) \langle\, e^{\frac{1}{2\pi}\int d^2z \lambda^i(z)\phi_i(z)}\, \rangle\,, \tag{3.2}$$

in which the coupling constants λ^i in equation (2.1) for the partition function have been replaced by sources

$$\lambda^i(z) = \lambda^i + \delta\lambda^i(z) \tag{3.3}$$

where the $\delta\lambda^i(z)$ have compact support in z. The perturbation series is written

$$\ln Z(\lambda) = \ln Z(0) + \sum_{N=1}^{\infty} \ln Z_{(N)} \tag{3.4}$$

$$\ln Z_{(N)} = \frac{1}{N!}\int \frac{d^2z_1}{2\pi}\cdots\frac{d^2z_N}{2\pi}\; \lambda^{i_1}(z_1)\cdots\lambda^{i_N}(z_N)\; \langle\, \phi_{i_1}(z_1)\cdots\phi_{i_N}(z_N)\, \rangle_c \tag{3.5}$$

where the $\langle\, \cdots\, \rangle_c$ are the connected correlation functions in the reference CFT.

The connected correlation functions are distributions in the coordinates z_α (so that they can be integrated against the sources). Their singularities are on the diagonals, where some of the z_α coincide. Considered as functions of the coordinates z_α at non-coincident points, the correlation functions are unambiguously defined. The integrals of these functions can be singular at coincident points, so renormalization is required to define the correlation functions as distributions. The integrals must be cut off in some fashion, then counterterms added to the action so that each term of the perturbation series goes to a finite limit when the cutoff is removed. Different renormalization schemes are related by reparametrization of the λ^i. That is, different schemes produce different coordinate systems on the space of conformal field theories.

The expression for the curvature tensor in terms of the derivatives of the metric is especially simple in coordinates where the first derivatives of the metric vanish:

$$R_{ijkl} = \frac{1}{2}(\partial_k\partial_j g_{li} - \partial_k\partial_i g_{jl} - \partial_l\partial_j g_{ki} + \partial_l\partial_i g_{jk})\,. \tag{3.6}$$

Kutasov [1] pointed out that there is an especially simple renormalization scheme that gives such coordinates: the hard-sphere cutoff with minimal subtraction. The integrals of correlation functions are cut off by restricting them to the region $|z_\alpha - z_\beta| > \epsilon$, $\alpha \neq \beta$.

Minimal counterterms depending on ϵ are added to the action to cancel the divergences so that the limit $\epsilon \to 0$ becomes finite. The first derivatives of the metric are

$$\partial_k g_{ij} = \int \frac{d^2 z}{2\pi} \, \langle\, \phi_i(1)\, \phi_j(0)\, \phi_k(z)\, \rangle_c \,. \tag{3.7}$$

The three-point function vanishes identically at non-coincident points, by assumption 2 (the vanishing of the OPE coefficients). Minimal subtraction means that no finite counterterms are added to the action, so the three-point function vanishes as a distribution, so the first derivatives of the metric vanish. As Kutasov remarked, the second derivatives of the metric are

$$\partial_l \partial_j g_{ik} = \int \frac{d^2 z_1}{2\pi} \frac{d^2 z_2}{2\pi} \, \langle\, \phi_i(1)\, \phi_k(0)\, \phi_l(z_1) \phi_j(z_2)\, \rangle_c \tag{3.8}$$

so the curvature tensor is given by a sum of double integrals of four-point functions. We take the calculation one step further. Conformal invariance implies that the four-point function depends, at non-coincident points, only on one argument, the cross-ratio

$$\eta = (1, z_1; z_2, 0) = \frac{(1-z_2) z_1}{z_1 - z_2} \,, \tag{3.9}$$

so we can perform one of the integrals explicitly, reducing the curvature formula to a single integral of the four-point function. The calculation is complicated by the need for regularization.

4 The conformal anomaly

We find it convenient to calculate the curvature tensor by extracting the metric from the integrated conformal anomaly

$$\mu \frac{\partial}{\partial \mu} \ln Z(\lambda) = \int d^2 z \, \langle \Theta(z) \rangle \,. \tag{4.1}$$

Here μ is the 2-d scale and $\Theta(z)$ is the trace of the stress-energy tensor. As a local field, $\Theta(z)$ can be expanded in a basis of scaling fields of real dimensions ≥ 0 and integer spins. $\Theta(z)$ has canonical dimension 2 and spin 0, and the sources $\lambda^i(z)$ are dimensionless, so the fields that contribute to $\Theta(z)$ have dimensions 0, 1, and 2. The only scaling field of dimension 0 is the identity. Thus the general form of the expectation value of $\Theta(z)$ is [5, 7]

$$\begin{aligned} 2\pi \langle\, \Theta(z)\, \rangle_{\lambda,c} = \beta^I(\lambda) \langle\, \phi_I(z)\, \rangle_{\lambda,c} + C_i^m(\lambda) \bar\partial \lambda^i \langle\, J_m(z)\, \rangle_{\lambda,c} + C_i^{\bar m}(\lambda) \partial \lambda^i \langle\, \bar J_{\bar m}(\bar z)\, \rangle_{\lambda,c} \\ + \partial_\mu \left[w_i(\lambda) \partial^\mu \lambda^i \right] - \frac{1}{8} g_{ij}(\lambda) \partial_\mu \lambda^i \partial^\mu \lambda^j \end{aligned} \tag{4.2}$$

where the ϕ_I are the dimension ≤ 2, spin 0 fields in the reference CFT and the $J_m(z)$, $\bar J_{\bar m}(\bar z)$ are the dimension 1, spin 1 (chiral) currents in the reference CFT. The coefficients on the right hand side are local functionals of the sources, of appropriate dimension and spin. The last two terms on the right hand side are proportional to the identity field. We will check later the appearance of the metric g_{ij} in the last term, and its coefficient. The last four terms on the right hand side comprise the *conformal anomaly* (in a flat 2-d geometry).

The beta functions $\beta^I(\lambda)$ of course vanish identically on a family of CFT's so the first term on the right hand side does not occur. But our assumptions only require that the $\beta^I(\lambda)$ vanish through second order. To calculate the curvature tensor, we will expand equation (4.2) to fourth order in the sources. The fourth derivative of $\beta^I(\lambda)$ will multiply a one-point function, which vanishes. The third derivative of $\beta^I(\lambda)$ will be symmetric in the three indices, so cannot contribute to the curvature tensor. So we can ignore the first term on the right hand side of (4.2). To avoid cluttering the calculations, we will take the third derivatives of $\beta^I(\lambda)$ to be zero. As we have argued, the result for the curvature tensor is not affected.

Equation (4.2) implies the OPE in the reference CFT of the form

$$T(z)\,\phi_i(0) \sim \frac{1}{z^3} C_i^{\bar{m}}(0)\bar{J}_{\bar{m}}(0) + \cdots \tag{4.3}$$

where $T(z)$ is the usual holomorphic component of the stress-energy tensor. Such a term is forbidden by global conformal invariance and unitarity. Therefore $C_i^{\bar{m}}(0) = 0$, and similarly $C_i^m(0) = 0$. In appendix A we show that the first derivatives $C_{ij}^m = \partial_i C_j^m(0)$ and $C_{ij}^{\bar{m}} = \partial_i C_j^{\bar{m}}(0)$ appear as operator product coefficients

$$\phi_i(z)\,\phi_j(0) \sim |z|^{-4}\left[z\, C_{ij}^m J_m(0) + \bar{z}\, C_{ij}^{\bar{m}} \bar{J}_{\bar{m}}(0)\right] \tag{4.4}$$

and thus vanish by assumption 3. This is enough to show that the second and third terms on the right hand side of (4.2) do not contribute to the curvature calculation. The fourth derivatives of $C_i^m(\lambda)$ multiply $\langle\, J_m \,\rangle_{0,c}$ which vanishes. The third derivatives multiply two-point functions $\langle\, J_m \phi_k \,\rangle_{0,c}$ which vanish. Finally, the second derivatives of $C_i^m(\lambda)$ multiply three-point functions $\langle\, J_m \phi_j \phi_k \,\rangle_{0,c}$ which vanish by assumption 3. The same holds for $C_i^{\bar{m}}(\lambda)$.

The fourth term in (4.2) is a total derivative so we can write

$$\mu \frac{\partial}{\partial \mu} \ln Z(\lambda) = \int d^2 z\, \langle\, \Theta(z) \,\rangle_{\lambda,c} = -\int \frac{d^2 z}{2\pi} \frac{1}{8}\, g_{ij}(\lambda)\, \partial_\mu \lambda^i \partial^\mu \lambda^j + \cdots \tag{4.5}$$

where the omitted terms make no contribution to the curvature tensor.

The tensor $g_{ij}(\lambda)$ in (4.5) is the Zamolodchikov metric (3.1). This is derived by noting that, with the hard-sphere regularization, the divergent part of

$$\frac{1}{2} \int \frac{d^2 z_1}{2\pi} \frac{d^2 z_2}{2\pi}\, \theta(|z_1 - z_2| - \epsilon)\; \lambda^i(z_1) \lambda^j(z_2) \langle\, \phi_i(z_1)\, \phi_j(z_2) \,\rangle_c \tag{4.6}$$

is cancelled by the counterterms

$$\Delta S = \int \frac{d^2 z}{2\pi} \left[\epsilon^{-2} \frac{1}{4} g_{ij} \lambda^i \lambda^j + \ln(\mu\epsilon) \frac{1}{8}\, g_{ij}\, \partial_\mu \lambda^i \partial^\mu \lambda^j\right] \tag{4.7}$$

so

$$\mu \frac{\partial}{\partial \mu} \ln Z(\lambda) = -\int \frac{d^2 z}{2\pi} \frac{1}{8}\, g_{ij}\, \partial_\mu \lambda^i \partial^\mu \lambda^j \tag{4.8}$$

to second order in the sources $\lambda^i(z)$. This local calculation works as well in any nearby conformal field theory, so the integrated anomaly must be as in equation (4.5). The equation does not depend on the renormalization scheme because no finite counterterms can affect it.

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5 The curvature tensor

The second derivatives of the metric are now found by expanding the anomaly to fourth order in the λ^i,

$$\mu\frac{\partial}{\partial\mu}\ln Z_{(4)} = -\int \frac{d^2z}{2\pi}\frac{1}{16}\partial_k\partial_l\, g_{ij}\,\lambda^k\lambda^l\partial_\mu\lambda^i\partial^\mu\lambda^j \tag{5.1}$$

where the fourth order term in the conformal perturbation series is

$$\ln Z_{(4)} = \frac{1}{4!}\int\prod_{\alpha=1}^{4}\frac{d^2z_\alpha}{2\pi}\,\langle\,\prod_{\alpha=1}^{4}\lambda^i(z_\alpha)\phi_i(z_\alpha)\,\rangle_c\,. \tag{5.2}$$

Changing integration variables to $y_\alpha = z_\alpha - z$ with $z = \sum_\alpha z_\alpha$, then expanding each $\lambda^{i_\alpha}(z+y_\alpha)$ in powers of the y_α, keeping the terms containing two derivatives of the sources, gives

$$\mu\frac{\partial}{\partial\mu}\ln Z_{(4)} = -\int d^2z\;\lambda^{i_2}\lambda^{i_3}\partial_\mu\lambda^{i_1}\partial^\mu\lambda^{i_4}\;\frac{1}{16}\int\prod_{\alpha=1}^{4}\frac{d^2y_\alpha}{2\pi}\;\delta^2\left(\frac{1}{4}\sum y_\alpha\right)\;|y_1-y_4|^2 \tag{5.3}$$

$$\times\,\mu\frac{\partial}{\partial\mu}\langle\,\phi_{i_1}(y_1)\phi_{i_2}(y_2)\phi_{i_3}(y_3)\phi_{i_4}(y_4)\,\rangle_c \tag{5.4}$$

from which we can read off the second derivatives of the metric

$$\partial_{i_2}\partial_{i_3}g_{i_1i_4} = \int\prod_{\alpha=1}^{4}\frac{d^2y_\alpha}{2\pi}\;2\pi\delta^2\left(\frac{1}{4}\sum y_\alpha\right)\;|y_1-y_4|^2 \tag{5.5}$$

$$\times\,\mu\frac{\partial}{\partial\mu}\langle\,\phi_{i_1}(y_1)\phi_{i_2}(y_2)\phi_{i_3}(y_3)\phi_{i_4}(y_4)\,\rangle_c\,. \tag{5.6}$$

Substituting in equation (3.6), we obtain

$$R_{i_1i_2i_3i_4} = \frac{1}{2}\int\prod_{\alpha=1}^{4}\frac{d^2y_\alpha}{2\pi}\;2\pi\delta^2\left(\frac{1}{4}\sum y_\alpha\right) \tag{5.7}$$

$$\times\left(|y_1-y_4|^2 - |y_2-y_4|^2 - |y_1-y_3|^2 + |y_2-y_3|^2\right) \tag{5.8}$$

$$\times\,\mu\frac{\partial}{\partial\mu}\langle\,\phi_{i_1}(y_1)\phi_{i_2}(y_2)\phi_{i_3}(y_3)\phi_{i_4}(y_4)\,\rangle_c\,. \tag{5.9}$$

Changing variables from y_α to $x_\alpha = y_\alpha - y_4$, $\alpha = 1,2,3$, and integrating over y_4, we obtain

$$R_{i_1i_2i_3i_4} = \int\prod_{\alpha=1}^{3}\frac{d^2x_\alpha}{2\pi}\;\left[(x_1-x_2)\cdot x_3\right]\mu\frac{\partial}{\partial\mu}\langle\,\phi_{i_1}(x_1)\phi_{i_2}(x_2)\phi_{i_3}(x_3)\phi_{i_4}(0)\,\rangle_c \tag{5.10}$$

where we write

$$u\cdot v = \frac{1}{2}\left(\bar u v + u\bar v\right) = \mathbf{Re}(\bar u v)\,. \tag{5.11}$$

The scale derivative of the four-point correlation function is the fourth variation of the integrated anomaly with respect to the sources. By the arguments of the previous section, the scale derivative vanishes away from the diagonal $x_1 = x_2 = x_3 = 0$. Thus the region of integration can be restricted to any region that includes the diagonal.

To construct the renormalized four-point function we use a version of analytic regularization. We define the regulated N-point function

$$G_s(\mathbf{z}) = \mu^{Ns} K_s(\mathbf{z}) \langle \phi_{i_1}(z_1) \dots \phi_{i_n}(z_n) \rangle_c \,. \tag{5.12}$$

where

$$K_s(\mathbf{z}) = \prod_{\alpha<\beta} |z_\alpha - z_\beta|^{\frac{2s}{N-1}} \,. \tag{5.13}$$

The crucial point of this definition is that the regulated fields ϕ_i have scaling dimension $2-s$, as in dimensional regularization of lagrangian quantum field theory. Then $\mu^s \phi_i$ has dimension 2 so we still have the canonical scaling relation

$$\left(\mu \frac{\partial}{\partial \mu} - \sum_\alpha z_\alpha \cdot \frac{\partial}{\partial z_\alpha} - 2N \right) G_s(\mathbf{z}) = 0 \,. \tag{5.14}$$

For $\mathbf{Re}\, s > 1$, the regulated correlation functions $G_s(\mathbf{z})$ are nonsingular distributions in the coordinates z_α. They are holomorphic functions of the regularization parameter s which analytically continue to meromorphic functions of s. The renormalized correlation functions are obtained by subtracting poles at $s=0$ and taking the limit $s \to 0$

$$\langle \phi_{i_1}(z_1) \dots \phi_{i_n}(z_n) \rangle_c = \lim_{s\to 0} \left[G_s(\mathbf{z}) - \Delta G_s(\mathbf{z}) \right] \tag{5.15}$$

where the counterterm $\Delta G_s(\mathbf{z})$ contains poles at $s=0$ and is independent of μ. Thus

$$\mu \frac{\partial}{\partial \mu} \langle \phi_{i_1}(z_1) \dots \phi_{i_n}(z_n) \rangle_c = \lim_{s\to 0} \mu \frac{\partial}{\partial \mu} G_s(\mathbf{z}) \,. \tag{5.16}$$

Equation (5.10) becomes

$$R_{i_1 i_2 i_3 i_4} = \lim_{s\to 0} \int \prod_{\alpha=1}^{3} \frac{d^2 x_\alpha}{2\pi} \; [(x_1 - x_2)\cdot x_3]\, \mu \frac{\partial}{\partial \mu} G_s(\mathbf{x}) \tag{5.17}$$

where $G_s(\mathbf{x})$ now stands for the regulated four-point function

$$G_s(\mathbf{x}) = \mu^{4s} K_s(\mathbf{x}) \langle\, \phi_{i_1}(x_1) \phi_{i_2}(x_2) \phi_{i_3}(x_3) \phi_{i_4}(0) \,\rangle_c \tag{5.18}$$

$$K_s(\mathbf{x}) = |x_1 x_2 x_3 (x_3 - x_1)(x_3 - x_2)(x_1 - x_2)|^{\frac{2s}{3}} \,. \tag{5.19}$$

By (5.14),

$$R_{i_1 i_2 i_3 i_4} = \lim_{s\to 0} \int \prod_{\alpha=1}^{3} \frac{d^2 x_\alpha}{2\pi} \; \sum_\alpha \frac{\partial}{\partial x_\alpha} \cdot \left(x_\alpha \; [(x_1 - x_2)\cdot x_3]\, G_s(\mathbf{x}) \right) . \tag{5.20}$$

Since the integral in (5.20) vanishes off the diagonal, we can introduce — without affecting the result — a factor $B(\mathbf{x})$ in the integrand that equals 1 in a neighborhood of $(0,0,0,0)$ and drops off sufficiently fast at infinity. We pick

$$B(\mathbf{x}) = e^{-\epsilon^2 \|\mathbf{x}\|^2} \tag{5.21}$$

with

$$\|\mathbf{x}\|^2 = |x_3|^2 + |x_1|^2 + |x_2 - x_3|^2\,. \tag{5.22}$$

Integrating by parts in (5.20) we obtain

$$R_{i_1i_2i_3i_4} = \lim_{s\to 0}\int \prod_{\alpha=1}^{3}\frac{d^2x_\alpha}{2\pi}\, B(\mathbf{x})\sum_\alpha \partial_\alpha\cdot\left(x_\alpha\,[(x_1-x_2)\cdot x_3]\,G_s(\mathbf{x})\right) \tag{5.23}$$

$$= \lim_{s\to 0}\int \prod_{\alpha=1}^{3}\frac{d^2x_\alpha}{2\pi}\,(-DB)(\mathbf{x})\,[(x_1-x_2)\cdot x_3]\,G_s(\mathbf{x}) \tag{5.24}$$

where

$$DB(\mathbf{x}) = \sum_\alpha x_\alpha\cdot\partial_\alpha B(\mathbf{x})\,. \tag{5.25}$$

The regulated correlation function depends only on the correlation function at non-coincident points, which is invariant under global conformal transformations, so we can rewrite the last formula as

$$R_{i_1i_2i_3i_4} = \lim_{s\to 0}\int \prod_{\alpha=1}^{3}\frac{d^2x_\alpha}{2\pi}\,(-DB)(\mathbf{x})$$
$$\times\,[(x_1-x_2)\cdot x_3]\,|x_1(x_3-x_2)|^{-4}\,K_s(\mathbf{x})G(1,\eta,\infty,0) \tag{5.26}$$

where

$$G(1,\eta,\infty,0) = \langle\phi_{i_1}(1)\phi_{i_2}(\eta)\phi_{i_3}(\infty)\phi_{i_4}(0)\rangle_c \tag{5.27}$$

and

$$\eta = \frac{(x_3-x_1)x_2}{(x_3-x_2)x_1} \tag{5.28}$$

is the cross-ratio. We have dropped the factor μ^{4s} from the regulated four-point function because the limit $s\to 0$ is finite.

We now have the curvature formula as a single integral

$$R_{i_1i_2i_3i_4} = \lim_{s\to 0}\int \frac{d^2\eta}{2\pi}\, f_s(\eta)G(1,\eta,\infty,0) \tag{5.29}$$

with

$$f_s(\eta) = \int \prod_{\alpha=1}^{3}\frac{d^2x_\alpha}{2\pi}\, 2\pi\delta^2\left(\eta - \frac{(x_3-x_1)x_2}{(x_3-x_2)x_1}\right)(-DB)(\mathbf{x})$$
$$\times\,[(x_1-x_2)\cdot x_3]\,|x_1(x_3-x_2)|^{-4}\,K_s(\mathbf{x})\,. \tag{5.30}$$

Changing the variables of integration to $u_1 = x_1/x_3$, $u_2 = x_2/x_3$ and x_3 and using (5.21) we can perform the integration over x_3 in (5.30) to get

$$f_s(\eta) = \frac{1}{2\pi}\int d^2u_1\, d^2u_2\,\delta^2\left(\eta - \frac{u_1^{-1}-1}{u_2^{-1}-1}\right)$$
$$\times\,\mathbf{Re}(u_1-u_2)\,|u_1(1-u_2)|^{-4}\,K_s(\mathbf{u})\Gamma(2s+1)\epsilon^{-4s}\|\mathbf{u}\|^{-4s} \tag{5.31}$$

where

$$\|\mathbf{u}\|^2 = 1 + |u_1|^2 + |1-u_2|^2\,, \tag{5.32}$$

$$K_s(\mathbf{u}) = |u_1 u_2 (1-u_1)(1-u_2)(u_1-u_2)|^{\frac{2s}{3}}\ . \tag{5.33}$$

Performing a further change of variables

$$v = \frac{1}{u} - 1\,, \qquad w = \left(\frac{1}{u_2} - 1\right)^{-1} \tag{5.34}$$

and using the delta function to integrate out w we obtain

$$f_s(\eta) = \Gamma(2s+1)\epsilon^{-4s}|\eta(\eta-1)|^{\frac{2s}{3}}\ \mathbf{Re}\ g_s(\eta)\,, \tag{5.35}$$

$$g_s(\eta) = (\eta-1)\int \frac{d^2v}{2\pi}\ |v|^{-2}\left[(1-v)(\eta-v)v^{-1}\right]^{-1}\left|(1-v)\,(\eta-v)\,v^{-1}\right|^{-2s}\|\mathbf{u}\|^{-4s}\,, \tag{5.36}$$

$$\|\mathbf{u}\|^2 = |1-v|^{-2} + |1-\eta^{-1}v|^{-2} + 1\,. \tag{5.37}$$

The function $f_s(\eta)$ is integrated against the four-point function $G(1,\eta,\infty,0)$ which has singularities at $\eta = 0, 1, \infty$. Away from those three points, as we will see shortly,

$$\lim_{s\to 0} f_s(\eta) = -\ln|\eta|\,. \tag{5.38}$$

Near the singular points $0, 1, \infty$, we have to perform the integral with $s > 0$ then take the limit $s \to 0$. Because the function $f_s(\eta)$ is so complicated, it is not immediately obvious how the regularization works. We will make the regularizing effect of $f_s(\eta)$ explicit by analyzing the integral in the immediate neighborhood of the singular points. We will then replace the regularization by $f_s(\eta)$ by an equivalent but much simpler prescription which uses the hard-sphere regularization.[2]

To see how the singularities are regularized we need to know the behaviour of $f_s(\eta)$ near $\eta = 0, 1, \infty$. The analysis is somewhat tedious but straightforward.[3] We find the following expressions

$$f_s(\eta) = \begin{cases} |\eta|^{\frac{2s}{3}}\ \left[-A_s^{(0)}(\eta)\ln|\eta| + B_s^{(0)}(\eta)\right] & |\eta| < 1 \\ |1-\eta|^{\frac{2s}{3}}\mathbf{Re}\ \left[(1-\eta)A_s^{(1)}(1-\eta)\right] & |1-\eta| < 1 \\ |\eta|^{-\frac{2s}{3}}\ \left[-A_s^{(\infty)}(\eta^{-1})\ln|\eta| + B_s^{(\infty)}(\eta^{-1})\right] & 1 < |\eta| \end{cases} \tag{5.39}$$

where $A_s^{(0,\infty)}$ and $B_s^{(0,\infty)}$ are real-valued, $A_s^{(1)}$ is complex valued, and all five functions are real-analytic in η in the appropriate domains, for $s > 0$. All are regular in s for $\mathbf{Re}\,s > -\frac{1}{2}$. At $s = 0$ we have

$$A_0^{(0)}(\eta) = A_0^{(\infty)}(\eta) = 1\,, \quad B_0^{(0)}(\eta) = B_0^{(\infty)}(\eta) = 0\,, \quad A_0^{(1)}(1-\eta) = -\eta^{-1}\ln(1-\eta)\,. \tag{5.40}$$

[2] We did not use hard-sphere regularization from the beginning because, as we will see later for boundary CFT, the reduction of the curvature formula to a single integral using hard-sphere regularization is a complicated calculation.

[3] It is advantageous to split the regions of v-integration into 3 parts. Thus for $\eta \to 0$ we can take $0 \le |v| \le |\eta|/\rho$, $|\eta|/\rho \le |v| \le \rho$, $\rho \le |v| < \infty$ where ρ is any real number such that $|\eta| < \rho < 1$. For η tending to ∞ and 1 same type of splitting is obtained after first changing the variables as $v \to v^{-1}$ and $v \to 1-v$ respectively.

Singularities of the four-point function $G(1, \eta, \infty, 0)$ arise from three sources. At $\eta = 0$, the relevant spin 0 fields give singularities which go as $|\eta|^{\Delta-4}$ with $0 < \Delta < 2$. While we excluded chiral spin 1 fields from the OPE's, the non-chiral spin 1 fields contribute divergences $|\eta|^{\delta-4}\eta$ with $0 < \delta < 1$. Finally, the fields of spin 2 and dimension 2 contribute singularities that go as $|\eta|^{-4}\eta^2$. Using the expressions (5.39) we find that all of these singularities are regularized when multiplied by $f_s(\eta)$ as long as $\mathbf{Re}\, s > 0$. For the contribution of a relevant field, we find

$$\frac{1}{2\pi}\int_{|\eta|<a} d^2\eta\, f_s(\eta)\, |\eta|^{\Delta-4} = \frac{-a^{\Delta-2}\ln a}{\Delta-2} + \frac{a^{\Delta-2}}{(\Delta-2)^2} + \cdots \tag{5.41}$$

where the ommitted terms vanish as $a \to 0$. The contributions of spin 1 and spin 2 fields vanish by rotation invariance. We thus see that the regularization by $f_s(\eta)$ is equivalent to the hard-sphere cutoff plus minimal counterterms of the form (5.41) for each relevant scalar field that occurs in the $\phi_{i_2}(\eta)\,\phi_{i_4}(0)$ OPE. In the absence of relevant scalar fields in the OPE, there are no counterterms at $\eta = 0$ and we simply get the principal value prescription.

The analysis at $\eta = \infty$ is exactly the same. The hard-sphere cutoff is $|\eta| < a^{-1}$. The integral near ∞, over the region $|\eta| \geq a^{-1}$, contributes minimal counterterms for each relevant scalar in the $\phi_{i_2}(\eta)\phi_{i_3}(\infty)$ OPE.

Finally, the extra factor $\eta - 1$ in the asymptotics of $f_s(\eta)$ as $\eta \to 1$ means that the integral over the region $|\eta - 1| \leq a$ is finite in the limit $a \to 0$, so we just have the principal value prescription at $\eta = 1$.

We have obtained

$$\begin{aligned} R_{i_1 i_2 i_3 i_4} &= \mathrm{RV}\int \frac{d^2\eta}{2\pi}\,(-\ln|\eta|)\,\langle\, \phi_{i_1}(1)\phi_{i_2}(\eta)\phi_{i_3}(\infty)\phi_{i_4}(0)\,\rangle_c \\ &= \lim_{a\to 0}\Bigg[\int\limits_{\substack{a<|\eta|<a^{-1}\\ a<|1-\eta|}} \frac{d^2\eta}{2\pi}\,(-\ln|\eta|)\,\langle\, \phi_{i_1}(1)\phi_{i_2}(\eta)\phi_{i_3}(\infty)\phi_{i_4}(0)\,\rangle_c + \Delta R_{i_1 i_2 i_3 i_4}(a)\Bigg] \end{aligned} \tag{5.42}$$

where $\Delta R_{i_1 i_2 i_3 i_4}(a)$ are the minimal counterterms due to relevant scalars, as explained above.

This formula was obtained using a regularization in which the first derivatives of the metric vanish. Our final formula (5.42) depends only on the values of the four-point functions at finite separations, therefore it transforms covariantly as a 4-tensor. Therefore (5.42) is coordinate-independent.

It is slightly nontrivial to check the symmetry properties of $R_{i_1 i_2 i_3 i_4}$ given by (5.42) and the first Bianchi identity. Formally they follow directly from invariance of the four-point function under the conformal transformations that permute $0, 1, \infty$, but the regularization is not manifestly conformally invariant. Under our assumptions 1-3, it turns out that that the regularization does not spoil the global conformal symmetries.

6 Two-dimensional torus example

To check the curvature formula we look at the moduli space of the two-dimensional torus CFT. This model can be described in terms of a free complex bosonic field $X(z, \bar{z})$ subject

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to identifications

$$X \sim X + 2\pi\,, \quad X \sim X + 2\pi i \tag{6.1}$$

The action is

$$S = \int \frac{d^2 z}{2\pi i}(\tau \partial X \bar{\partial} X^* - \bar{\tau}\bar{\partial} X \partial X^*) \tag{6.2}$$

where X^* is the complex conjugate field and τ is the coupling constant that specifies the Kahler form on the target space two-torus. We are considering the family of CFT's parametrized by τ. For simplicity we hold fixed the target space complex structure.

The propagator is

$$\langle X^*(z,\bar{z}) X(0)\rangle = -\frac{1}{\operatorname{Im}\tau} \ln |z|^2\,. \tag{6.3}$$

The variation of the Kahler modulus τ is described by the action variation

$$\delta S = -\int \frac{d^2 z}{2\pi}(\delta\tau \phi_\tau + \delta\bar{\tau}\phi_{\bar{\tau}}) \tag{6.4}$$

where

$$\phi_\tau = -i\partial X \bar{\partial} X^*\,, \qquad \phi_{\bar{\tau}} = i\bar{\partial} X \partial X^*\,. \tag{6.5}$$

The two-point function

$$\langle \phi_{\bar{\tau}}(z,\bar{z})\phi_\tau(0)\rangle = (\operatorname{Im}\tau)^{-2}|z|^{-4} \tag{6.6}$$

gives the Zamolodchikov metric

$$ds^2 = g_{\bar{\tau}\tau}|d\tau|^2 + g_{\tau\bar{\tau}}|d\tau|^2 = 2(\operatorname{Im}\tau)^{-2}|d\tau|^2\,, \qquad g_{\bar{\tau}\tau} = g_{\tau\bar{\tau}} = (\operatorname{Im}\tau)^{-2}\,. \tag{6.7}$$

The curvature tensor is

$$R_{ijkl} = \frac{1}{2}(g_{il}g_{jk} - g_{ik}g_{jl})\,, \quad R_{\bar{\tau}\tau\bar{\tau}\tau} = \frac{1}{2} g_{\tau\bar{\tau}} g_{\bar{\tau}\tau}\,. \tag{6.8}$$

The coordinate τ thus describes the Poincare half-plane model of the 2d constant negative curvature space.

Using the connected four-point function

$$\langle \phi_{\bar{\tau}}(1)\phi_\tau(\eta)\phi_{\bar{\tau}}(\infty)\phi_\tau(0)\rangle_c = (\operatorname{Im}\tau)^{-4}\left[\frac{1}{(1-\eta)^2} + \frac{1}{(1-\bar{\eta})^2}\right] \tag{6.9}$$

we obtain from our general formula (1.1)

$$R_{\bar{\tau}\tau\bar{\tau}\tau} = -(\operatorname{Im}\tau)^{-4}\int \frac{d^2\eta}{2\pi} \ln|\eta| \left[\frac{1}{(1-\eta)^2} + \frac{1}{(1-\bar{\eta})^2}\right]. \tag{6.10}$$

Regularizing, as prescribed, by cutting a small circle around $\eta = 1$ and a large circle around the origin we obtain

$$R_{\bar{\tau}\tau\bar{\tau}\tau} = \frac{1}{2}(\operatorname{Im}\tau)^{-4} \tag{6.11}$$

matching (6.8).

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7 The space of conformal boundary conditions

We now turn to the case of boundary conformal field theories. A boundary conformal field theory (BCFT) is a conformal field theory on the disk with a conformally invariant boundary condition on the boundary circle. As in the bulk, the BCFT's are supposed to be unitary, with discrete spectrum, and to be invariant under the global conformal group.

The disk can be mapped conformally to the upper half-plane with the boundary becoming the projective line — the real axis plus the point at infinity. We find it convenient to calculate in the coordinate $x = \tan(\theta/2)$, $-\pi \leq \theta < \pi$, on the projective line. For purposes of regularization, we use the metric transported from the unit circle

$$(ds)^2 = (d\theta)^2 = \rho(x)^2 (dx)^2\,, \qquad \rho(x) = \frac{1}{1+x^2}\,, \tag{7.1}$$

because it treats all points on the boundary uniformly, including the point at $x = \infty$.

We are studying smooth families of boundary CFT's for a given, fixed bulk CFT. Such a family — that is, a smooth family of conformal boundary conditions for the given CFT — is parameterized by dimensionless coupling constants λ^a which couple to local, dimension 1 boundary fields $\psi_a(x)$ so that

$$\frac{\partial}{\partial \lambda^a} \langle \mathcal{O} \rangle = \int dx\, \langle \psi_a(x)\, \mathcal{O} \rangle \tag{7.2}$$

where $\mathcal{O}$ stands for an arbitrary product of local operators. The natural metric[4] on the family of BCFT's is read off from the two-point function

$$\langle \psi_a(x_1) \psi_b(x_2) \rangle = g_{ab} (x_1 - x_2)^{-2} \tag{7.3}$$

or

$$g_{ab} = \langle \psi_a(0) \psi_b(\infty) \rangle\,, \qquad \psi_b(\infty) = \lim_{x\to\infty} x^2 \psi_b(x)\,. \tag{7.4}$$

We choose a reference BCFT satisfying assumptions similar to those made for the reference bulk CFT:

Assumption 1b *The $\psi_a(x)$ are dimension 1 boundary fields.*

The remaining two assumptions have to do with the $\psi_a\, \psi_b$ OPE, whose singular part has the following general form for $x > 0$

$$\psi_a(x)\, \psi_b(0) \sim x^{-2} g_{ab} \mathbf{1} + x^{-1} \sum_{\tilde{c}} C_{ab}^{\tilde{c}} \psi_{\tilde{c}}(0) + \sum_{c'} x^{\Delta_{c'}-2} C_{ab}^{c'} \psi_{c'}(0)\,. \tag{7.5}$$

[4] A metric on the space of not-necessarily-conformal boundary conditions was defined in [8, 9] in connection with the proof of the g-theorem [9, 10],

$$g_{ab} = \frac{2}{\pi} \int_0^{2\pi} d\theta\, \sin^2\left(\frac{\theta - \theta'}{2}\right) \langle \psi_a(\theta) \psi_b(\theta') \rangle\,.$$

For conformal boundary conditions, this agrees with (7.4).

Here, the $\psi_{\tilde{c}}$ are all the dimension 1 fields, which include our perturbations ψ_c. The $\psi_{c'}$ are all the relevant boundary fields (except for the identity $\mathbf{1}$) — the fields of scaling dimensions $\Delta_{c'} < 1$. The OPE for $x < 0$ is, by translation invariance,

$$\psi_a(x)\,\psi_b(0) \sim \psi_b(-x)\,\psi_a(0)\,. \tag{7.6}$$

Assumption 2b *The OPE coefficients $C^{\tilde{c}}_{ab}$ are antisymmetric: $C^{\tilde{c}}_{ab} = -C^{\tilde{c}}_{ba}$.*

Assumption 3b *The OPE coefficients $C^{c'}_{ab}$ are symmetric, $C^{\tilde{c}}_{ab} = C^{\tilde{c}}_{ba}$, for all dimension 0 fields $\psi_{c'}$.*

Assumptions 1b and 2b imply that the beta functions for the couplings λ^a vanish at least through the second order. Assumption 1b excludes boundary perturbations by derivative fields. Assumption 3b parallels bulk assumption 3. Dimension 0 boundary fields other than the identity arise when the BCFT has degenerate ground states (described in string theory by Chan-Paton indices). The dimension 0 fields act as charges which generate global symmetries, mixing the degenerate sectors of the BCFT. Assumption 3b means that the perturbations commute with these charges, that there are no boundary condition changing perturbations. In appendix A we show that, if there is a charged perturbation, then a certain linear combination of the ψ_a becomes a derivative field at first order in the symmetry breaking perturbation. Therefore, as in the bulk, our assumptions systematically exclude derivative operators.

8 The boundary curvature formula

We use a hard sphere cutoff on the boundary, renormalizing the correlation functions by minimal subtraction. The cutoff is $d(x_1, x_2) > \epsilon'$ where the distance function is carried over from the unit circle

$$d(x_1, x_2) = |\theta_1 - \theta_2| = 2\left|\tan^{-1} x_1 - \tan^{-1} x_2\right|\,. \tag{8.1}$$

Thus the cutoff can be written equivalently

$$\left|\frac{x_1 - x_2}{1 + x_1 x_2}\right| \geq \epsilon\,, \tag{8.2}$$

where $\epsilon = \tan(\epsilon'/2)$. The cutoff function is

$$h_\epsilon(x_1, x_2) = \theta\left(\left|\frac{x_1 - x_2}{1 + x_1 x_2}\right| - \epsilon\right)\,. \tag{8.3}$$

In particular

$$h_\epsilon(x, 0) = \theta(|x| - \epsilon)\,, \qquad h_\epsilon(x, \infty) = \theta(|x|^{-1} - \epsilon)\,. \tag{8.4}$$

The regulated correlation functions are

$$\langle \psi_{a_1}(x_1) \cdots \psi_{a_N}(x_N) \rangle \prod_{\alpha<\beta} h_\epsilon(x_\alpha, x_\beta)\,. \tag{8.5}$$

The first derivatives of the metric at $\lambda^a = 0$ are given by

$$\partial_c g_{ab} = \int dx \, \langle \psi_c(x)\psi_a(0)\psi_b(\infty)\rangle \, . \tag{8.6}$$

It follows from assumption 2b that the cutoff integral vanishes,

$$\int\limits_{\epsilon\leq|x|\leq\epsilon^{-1}} dx \, \langle \psi_c(x)\psi_a(0)\psi_b(\infty)\rangle = 0 \, . \tag{8.7}$$

Therefore, since we are using minimal subtraction, no contact terms contribute to (8.6). We conclude that $\partial_c g_{ab} = 0$. The curvature tensor is given by

$$R_{abcd} = \frac{1}{2}(\partial_b\partial_c g_{ad} - \partial_a\partial_c g_{bd} - \partial_b\partial_d g_{ac} + \partial_a\partial_d g_{bc}) \, . \tag{8.8}$$

The regularized second derivatives of the metric are

$$(\partial_b\partial_c g_{ad})^\epsilon = \iint dx_1 dx_2 \, \langle \psi_b(x_1)\psi_c(x_2)\psi_a(0)\psi_d(\infty)\rangle_c H_\epsilon(x_1,x_2) \tag{8.9}$$

where

$$H_\epsilon(x_1,x_2) = h_\epsilon(x_1,x_2)h_\epsilon(x_1,0)h_\epsilon(x_1,\infty)h_\epsilon(x_2,0)h_\epsilon(x_2,\infty) \, . \tag{8.10}$$

The regularized curvature tensor R^ϵ_{abcd} is obtained by using the regularized derivatives of metric (8.9) in (8.8). The curvature tensor is then obtained as

$$R_{abcd} = \lim_{\epsilon\to 0}(R^\epsilon_{abcd} + \Delta R^\epsilon_{abcd}) \tag{8.11}$$

where ΔR^ϵ_{abcd} is the counterterm.

We write

$$R^\epsilon_{abcd} = \frac{1}{2}\left(\tilde{R}^\epsilon_{abcd} - \tilde{R}^\epsilon_{abdc}\right) \tag{8.12}$$

with

$$\tilde{R}^\epsilon_{abcd} = (\partial_b\partial_c g_{ad})^\epsilon - (\partial_a\partial_c g_{bd})^\epsilon \tag{8.13}$$

which is

$$\tilde{R}^\epsilon_{abcd} = \int_{\mathcal{R}^\epsilon} dx_1 dx_2 [\langle \psi_b(x_1)\psi_c(x_2)\psi_a(0)\psi_d(\infty)\rangle_c - \langle \psi_b(0)\psi_c(x_2)\psi_a(x_1)\psi_d(\infty)\rangle_c] \tag{8.14}$$

where the integration region is

$$\mathcal{R}^\epsilon = \{(x_1,x_2) : H_\epsilon(x_1,x_2) = 1\} \, . \tag{8.15}$$

Changing the variables of integration to $\chi = x_2/x_1$, $x = x_1$ and using the global conformal invariance of the correlation function we rewrite $\tilde{R}^\epsilon_{abcd}$ as

$$\tilde{R}^\epsilon_{abcd} = \int d\chi \, F_\epsilon(\chi)[\langle \psi_b(0)\psi_c(\chi)\psi_a(1)\psi_d(\infty)\rangle_c + \langle \psi_b(1)\psi_c(1-\chi)\psi_a(0)\psi_d(\infty)\rangle_c] \tag{8.16}$$

where

$$F_\epsilon(\chi) = \left[\int_{\mathcal{R}^\epsilon_+(1-\chi)} - \int_{\mathcal{R}^\epsilon_+(\chi)}\right]\frac{dx}{x} \tag{8.17}$$

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and

$$\mathcal{R}_{+}^{\epsilon}(\chi)=\{x: x>0,(x,\chi x)\in\mathcal{R}^{\epsilon}\}\,. \tag{8.18}$$

We show in appendix B.1 that $\tilde{R}^{\epsilon}_{abcd}=-\tilde{R}^{\epsilon}_{abdc}$, so

$$R^{\epsilon}_{abcd}=\tilde{R}^{\epsilon}_{abcd}\,. \tag{8.19}$$

We have now succeeded in expressing the regularized curvature tensor as a single integral. It is straightforward but very tedious to calculate $F_{\epsilon}(\chi)$. The result is a piecewise continuous function given in table 1. Away from the singular points $\chi=0,1,\infty$

$$\lim_{\epsilon\to 0}F_{\epsilon}(\chi)=-\ln|1-\chi^{-1}|\,. \tag{8.20}$$

Next we analyze the regularization. We write R^{ϵ}_{abcd} as the principal value regulated integral plus an error term,

$$R^{\epsilon}_{abcd}=R^{\rm PV}_{abcd}+E_{abcd} \tag{8.21}$$

where

$$R^{\rm PV}_{abcd}=\int d\chi F^{\rm PV}_{\epsilon}(\chi)[\langle\psi_b(0)\psi_c(\chi)\psi_a(1)\psi_d(\infty)\rangle_c+\langle\psi_b(1)\psi_c(1-\chi)\psi_a(0)\psi_d(\infty)\rangle_c] \tag{8.22}$$

with

$$F^{\rm PV}_{\epsilon}=-\ln|1-\chi^{-1}|\,\theta(\chi-\epsilon^2)\theta(1-\chi-\epsilon^2)\theta(\chi^{-1}-\epsilon^2) \tag{8.23}$$

and

$$E_{abcd}=\int d\chi\Delta F_{\epsilon}(\chi)[\langle\psi_b(0)\psi_c(\chi)\psi_a(1)\psi_d(\infty)\rangle_c+\langle\psi_b(1)\psi_c(1-\chi)\psi_a(0)\psi_d(\infty)\rangle_c] \tag{8.24}$$

where

$$\Delta F_{\epsilon}(\chi)=F_{\epsilon}(\chi)-F^{\rm PV}_{\epsilon}(\chi)\,. \tag{8.25}$$

The function $\Delta F_{\epsilon}(\chi)$ is also given in table 1.

The next step is to show that the combination of the error term E_{abcd} and the renormalization counterterm $\Delta R^{\epsilon}_{abcd}$ gives the minimal subtraction for principal value regularization. That is,

$$E_{abcd}+\Delta R^{\epsilon}_{abcd}=-(R^{\rm PV}_{abcd})_{\rm sing}+r(\epsilon) \tag{8.26}$$

where $(R^{\rm PV}_{abcd})_{\rm sing}$ is the singular part of $R^{\rm PV}_{abcd}$ and $r(\epsilon)\to 0$ as $\epsilon\to 0$.

The singularities of the four-point function are found using the OPE (7.5). For $\chi\to 0$,

$$\begin{aligned}\langle\,\psi_b(0)\psi_c(\chi)\psi_a(1)\psi_d(\infty)\,\rangle_c &\sim \chi^{-1}\sum_{\tilde{e}}C^{\tilde{e}}_{cb}C_{da\tilde{e}}\\ &\quad+\sum_{e'}|\chi|^{\Delta_{e'}-2}\left[C^{e'}_{cb}\theta(\chi)+C^{e'}_{bc}\theta(-\chi)\right]C_{dae'}\,, \end{aligned}\tag{8.27}$$

$$\begin{aligned}\langle\,\psi_b(0)\psi_c(1-\chi)\psi_a(1)\psi_d(\infty)\,\rangle_c &\sim \chi^{-1}\sum_{\tilde{e}}C^{\tilde{e}}_{ca}C_{db\tilde{e}}\\ &\quad+\sum_{e'}|\chi|^{\Delta_{e'}-2}\left[C^{e'}_{ac}\theta(\chi)+C^{e'}_{ca}\theta(-\chi)\right]C_{bde'}\,. \end{aligned}\tag{8.28}$$

For $\chi \to \infty$

$$\langle\, \psi_b(0)\psi_c(\chi)\psi_a(1)\psi_d(\infty)\,\rangle_c \sim \chi^{-1}\sum_{\tilde{e}} C_{dc\tilde{e}}C^{\tilde{e}}_{ab} + \sum_{e'} |\chi|^{-\Delta_{e'}} \left[C_{dce'}\theta(\chi) + C_{cde'}\theta(-\chi)\right] C^{e'}_{ab}\,. \tag{8.29}$$

We have defined the OPE coefficients with lowered indices by

$$C_{ab\tilde{c}} = \langle \psi_a(1)\psi_b(0)\psi_{\tilde{c}}(\infty)\rangle\,, \quad C_{abc'} = \langle \psi_a(1)\psi_b(0)\psi_{c'}(\infty)\rangle\,. \tag{8.30}$$

Using these expressions for the singular parts of the four-point function, we obtain

$$(R^{\rm PV}_{abcd})_{\rm sing} = \sum_{c'}\left[-\ln(\epsilon^2)\frac{(\epsilon^2)^{\Delta_{c'}-1}}{1-\Delta_{c'}} - \frac{(\epsilon^2)^{\Delta_{c'}-1}}{(1-\Delta_{c'})^2}\right] K^{c'}_{abcd} \tag{8.31}$$

where

$$K^{c'}_{abcd} = C^{c'}_{(ac)}C_{(bd)c'} - C^{c'}_{(ad)}C_{(bc)c'}\,, \qquad C^{c'}_{(ac)} = C^{c'}_{ac} + C^{c'}_{ca}\,. \tag{8.32}$$

Note that the dimension 1 fields make no contribution, because of the principal value regularization. We notice in calculating $(R^{\rm PV}_{abcd})_{\rm sing}$ that there is no contribution from $\chi = \infty$ because of the factor $-\ln|1-\chi^{-1}|$ in (8.23).

We next discuss the renormalization counterterm. In our regularization scheme the divergences from a pair of colliding insertions are obtained from the OPE (7.5)

$$\int dx_2\, \psi_a(x_2)\psi_b(x_1)h_\epsilon(x_1,x_2) \sim \sum_{c'} \frac{\epsilon^{\Delta_{c'}-1}}{1-\Delta_{c'}} C^{c'}_{(a,b)}\rho(x_1)^{1-\Delta_{c'}}\psi_{c'}(x_1) \tag{8.33}$$

where $\rho(x)$ is the scale factor of the metric on the boundary given in (7.1). This implies a counterterm for the action

$$\Delta S = \int dx\, \rho(x)\, \frac{1}{2}\sum_{c'}\frac{\epsilon^{\Delta_{c'}-1}}{1-\Delta_{c'}} C^{c'}_{(ab)}\lambda^a\lambda^b\, \rho(x)^{-\Delta_{c'}}\psi_{c'}(x)\,. \tag{8.34}$$

It is a standard calculation to find the contribution to the four-point functions of this counterterm to the action. We find

$$\Delta R_{abcd} = \sum_{c'}\frac{\epsilon^{\Delta_{c'}-1}}{1-\Delta_{c'}}\int_0^\infty dx\, (1+x^2)^{\Delta_{c'}-1}x^{-\Delta_{c'}}K^{c'}_{abcd} + \sum_{c'}\left(\frac{\epsilon^{\Delta_{c'}-1}}{1-\Delta_{c'}}\right)^2\left(-K^{c'}_{abcd}\right) \tag{8.35}$$

where $K^{c'}_{abcd}$ is given in (8.32).

The error term E_{abcd} defined in (8.24) can be evaluated explicitly up to terms tending to zero as $\epsilon \to 0$. We find

$$E_{abcd} = \sum_{c'}\ln(\epsilon^2)\frac{(\epsilon^2)^{\Delta_{c'}-1}}{1-\Delta_{c'}}K^{c'}_{abcd} + \sum_{c'}\frac{(\epsilon^2)^{\Delta_{c'}-1}}{(1-\Delta_{c'})^2}2K^{c'}_{abcd} + \sum_{c'}\frac{\epsilon^{\Delta_{c'}-1}}{1-\Delta_{c'}}\int_2^\infty du\, \left[-\partial_u A(u^{-2}) + 2u^{-1}\right]u^{\Delta_{c'}-1}\left(-K^{c'}_{abcd}\right) \tag{8.36}$$

where

$$A(u^{-2}) = -2\ln\left(\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{u^2}}\right). \tag{8.37}$$

The details of this computation are put into appendix B.3. The key point is that there are only singular terms in the limit $\epsilon \to 0$, no finite terms. Since the ΔR_{abcd} counterterm also contains only singular terms, we can conclude that

$$E_{abcd} + \Delta R_{abcd} = -(R^{\mathrm{PV}}_{abcd})_{\mathrm{sing}} \tag{8.38}$$

up to terms vanishing in the limit $\epsilon \to 0$. We can verify this equation explicitly using the identity

$$\int_2^\infty du\ \left[-\partial_u A(u^{-2}) + 2u^{-1}\right] u^{\Delta-1} = -\int_0^\infty dx\,(1+x^2)^{\Delta-1}x^{-\Delta}. \tag{8.39}$$

We thus arrive at the following formula for the curvature

$$\begin{aligned} R_{abcd} = \mathrm{RV}\int d\chi\,(-\ln|1-\chi^{-1}|)[\langle\psi_b(0)\psi_c(\chi)\psi_a(1)\psi_d(\infty)\rangle_c \\ + \langle\psi_b(1)\psi_c(1-\chi)\psi_a(0)\psi_d(\infty)\rangle_c] \end{aligned} \tag{8.40}$$

where the integral near $\chi = 0, 1$ is taken with principle value regularization and minimal subtraction. As we remarked before, no regularization is needed at $\chi = \infty$.

It should be noted that even when there are no relevant operators in the OPE, the integral in the curvature formula is still conditionally convergent around $\chi = 0, 1$ in general. The principal value prescription is still needed.

Changing integration variable to $\eta = 1 - \chi^{-1}$ and making a conformal transformation of the four-point functions, we obtain the boundary curvature formula stated in the Introduction

$$\begin{aligned} R_{abcd} = \mathrm{RV}\int_{-\infty}^{\infty} d\eta\ (-\ln|\eta|)\ [\langle\,\psi_a(1)\psi_b(\eta)\psi_c(\infty)\psi_d(0)\,\rangle_c \\ + \langle\,\psi_a(0)\psi_b(1-\eta)\psi_c(\infty)\psi_d(1)\,\rangle_c]\,. \end{aligned} \tag{8.41}$$

In this formula, no regularization is needed at $\eta = 1$. The change of variable $\eta = 1 - \chi^{-1}$ does not manifestly preserve the principal value regularization, so care is needed to check that the regularization is in fact preserved, given our assumptions.

As in the bulk, the boundary curvature formula depends only on the correlation functions at finite separation, so is coordinate independent.

9 D0 branes on group manifolds

As a check of the boundary curvature formula (8.41) we will consider the example of D0 brane boundary conditions on group manifolds. The bulk CFT is a WZW theory at level

k for a semisimple compact Lie group G. We pick a basis in the Lie algebra so that the corresponding currents $J^a(z)$ satisfy the OPE

$$J^a(z)J^b(w) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{ab}{}_c J^c(w)}{z-w} + \dots \tag{9.1}$$

where f^{ab}_c is the totally antisymmetric tensor of the Lie group structure constants. As a reference boundary condition we take the D0 brane located at the identity element. The boundary condition on a half plane glues the left and right components of the currents at the boundary as $J^a(x) = \bar{J}^a(x)$. As shown in [11] the boundary perturbation

$$\langle \exp \left(\int dx \sum_a \lambda^a J^a(x) \right) \dots \rangle \tag{9.2}$$

is exactly marginal for all values of the couplings λ^a. The λ^a parameterize the position $g(\lambda)$ of the D0 brane in the group manifold G. The corresponding boundary condition is

$$J^a(x) = (\mathrm{Ad}_g \bar{J})^a(x) \tag{9.3}$$

where Ad_g stands for the adjoint action of G on its Lie algebra. Since the moduli space is a homogeneous space it suffices to compute the curvature at a single point. We will first compute the curvature in terms of double integrals of distributional four-point functions, as in [1]. Then we check that the result agrees with our formula (8.41).

We will be calculating first and second derivatives of the metric which is given by the two-point function at finite separation

$$\langle\, J^c(x_3) J^d(x_4)\,\rangle = \frac{g_{cd}}{(x_3 - x_4)^2}\,. \tag{9.4}$$

The distributional correlation functions on the boundary are defined in appendix C. To find the first derivatives of the metric, we integrate the three-point function (C.17),

$$\int_{-\infty}^{+\infty} dx_1 \, \langle J^a(x_1) J^c(x_3) J^d(x_4) \rangle = 0 \tag{9.5}$$

at finite separation. Thus, in our coordinates,

$$\partial_a g_{cd/\lambda^a=0} = 0 \tag{9.6}$$

so we can use formula (8.8) to compute the curvature at the origin.[5]

Now we have to calculate the second derivatives of the metric. Integrating the distributional four-point function (C.43) once, we get

$$\int_{-\infty}^{+\infty} dx_1 \, \langle J^a(x_1) J^b(x_2) J^c(x_3) J^d(x_4) \rangle$$

[5]It is easy to see that any regularization of the three-point function of currents which preserves the group symmetry will have the same property.

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$$= \frac{k\pi^2}{3} f^{ac}{}_e f^{bed} \left(2\delta(x_{23}) \left[\frac{1}{x_{24}^2}\right] - \delta(x_{34}) \left[\frac{1}{x_{24}^2}\right] - \delta(x_{24}) \left[\frac{1}{x_{23}^2}\right] \right)$$

$$+ \frac{k\pi^2}{3} f^{ad}{}_e f^{bce} \left(-2\delta(x_{24}) \left[\frac{1}{x_{23}^2}\right] + \delta(x_{34}) \left[\frac{1}{x_{23}^2}\right] + \delta(x_{23}) \left[\frac{1}{x_{43}^2}\right] \right) \tag{9.7}$$

where the square brackets stand for the distributional regularization

$$\left[\frac{1}{x^2}\right] = -\partial_x \mathrm{PV}\left(\frac{1}{x}\right) . \tag{9.8}$$

Integrating one more time we obtain

$$\int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \, \langle J^a(x_1) J^b(x_2) J^c(x_3) J^d(x_4) \rangle = \frac{k\pi^2}{3} \left[\frac{1}{x_{34}^2}\right] (f^{ac}{}_e f^{bed} + f^{ad}{}_e f^{bec}) \tag{9.9}$$

so

$$\partial_a \partial_b g_{cd} = \frac{k\pi^2}{3} (f^{ac}{}_e f^{bed} + f^{ad}{}_e f^{bec}) . \tag{9.10}$$

From this expression it is easy to see that

$$\partial_a \partial_b g_{cd/\lambda^a=0} = \partial_c \partial_d g_{ab/\lambda^a=0} \tag{9.11}$$

and

$$[\partial_a \partial_b g_{cd} + \partial_a \partial_c g_{db} + \partial_a \partial_d g_{bc}]_{/\lambda^a=0} = 0 \tag{9.12}$$

implying that the distributional correlators defined in appendix C correspond to Riemann normal coordinates at the origin. From (9.10) we obtain

$$R_{abcd} = k\pi^2 f^{ab}{}_e f^{ced} . \tag{9.13}$$

The Killing metric g_{ab}^{Killing} on the group manifold has curvature tensor

$$R_{abcd}^{\text{Killing}} = \frac{1}{4} f^{ab}{}_e f^{ced} \tag{9.14}$$

so the metric on the space of conformal boundary conditions is

$$g_{ab} = 4\pi^2 k \, g_{ab}^{\text{Killing}} . \tag{9.15}$$

Next we check that our general curvature formula (8.41) gives the same result. The four-point function is

$$\langle J^b(\infty) J^c(\eta) J^a(0) J^d(1) \rangle_c = -\frac{k}{1-\eta} f^{ba}{}_e f^{ced} + \frac{k}{\eta} f^{bd}{}_e f^{cae} . \tag{9.16}$$

Substituting into (8.41) gives

$$R_{abcd} = -2k \left[f^{bd}{}_e f^{cae} I_1 + f^{ba}{}_e f^{ced} I_2 \right] \tag{9.17}$$

where

$$I_1 = \mathrm{PV} \int_{-\infty}^{\infty} d\eta \, \frac{\ln|\eta|}{\eta} = 0 \tag{9.18}$$

$$I_2 = -\mathrm{PV} \int_{-\infty}^{\infty} d\eta \, \frac{\ln|\eta|}{1-\eta} = \frac{\pi^2}{2} , \tag{9.19}$$

so (9.17) agrees with the direct computation (9.13).

10 Curvature formula and string theory effective action

Here we show that the curvature tensor (1.1) appears in the low energy action for massless scalars in string theory.

Suppose we have a CFT with integer central charge $c \leq 24$. The tensor product with $d = 26 - c$ free bosons X^μ is a bosonic string background. The massless scalar vertex operators are

$$V_i =: e^{iP\cdot X} :\ \phi_i\,, \quad P^2 = 0\,. \tag{10.1}$$

The Virasoro-Shapiro four-point amplitude for the massless scalars is

$$\delta^d\left(\sum P_\alpha\right)\mathcal{A}^{(4)}_{i_1 i_2 i_3 i_4}(s,t,u) = \int \frac{d^2\eta}{2\pi}\,\langle\, V_{i_1}(1)\; V_{i_2}(\eta)\; V_{i_3}(\infty)\; V_{i_4}(0)\,\rangle_c \tag{10.2}$$

with on-shell condition $s+t+u=0$. Substituting for the V_i and evaluating the free boson correlation functions, we obtain

$$\begin{aligned}\mathcal{A}^{(4)}_{i_1 i_2 i_3 i_4}(s,t,u) = &\left(\frac{tu\,(t+2)\,(u+2)}{8s\,(s+2)} g_{i_1 i_2} g_{i_3 i_4} + \frac{su\,(s+2)\,(u+2)}{8t\,(t+2)} g_{i_1 i_3} g_{i_2 i_4}\right.\\ &\left. + \frac{st\,(s+2)\,(t+2)}{8u\,(u+2)} g_{i_1 i_4} g_{i_2 i_3}\right) F(s,t,u)\\ &+ \int \frac{d^2\eta}{2\pi}\,|\eta|^{-t}|1-\eta|^{-s}\,\langle\,\phi_{i_1}(1)\;\phi_{i_2}(\eta)\;\phi_{i_3}(\infty)\;\phi_{i_4}(0)\,\rangle_c\end{aligned} \tag{10.3}$$

where

$$F(s,t,u) = \frac{\Gamma\left(1-\frac{1}{2}t\right)\Gamma\left(1-\frac{1}{2}s\right)\Gamma\left(1-\frac{1}{2}u\right)}{\Gamma\left(2+\frac{1}{2}t\right)\Gamma\left(2+\frac{1}{2}s\right)\Gamma\left(2+\frac{1}{2}u\right)}\,. \tag{10.4}$$

The usual assumption is that the low energy string scattering amplitudes come from an effective d-dimensional field theory action. The part that describes the self-interactions of the massless scalar fields $\Phi^i(X)$ is the d-dimensional non-linear sigma model

$$S_{\mathit{eff}} = \int d^dX\;\frac{1}{2} g_{ij}(\Phi)\partial_\mu\Phi^i\partial^\mu\Phi^j \tag{10.5}$$

where the Φ^i are coordinates on the space of CFT's and g_{ij} is the Zamolodchikov metric. As far as we know, this has never been proved. Accepting the assumption, the low energy limit of the four-point scattering amplitude due to self-interactions can be calculated by expanding the metric in Riemann normal coordinates,

$$S_{\mathit{eff}} = \int d^dX\;\left(\frac{1}{2}\delta_{ij}\partial_\mu\Phi^i\partial^\mu\Phi^j - \frac{1}{3}R_{ikjl}\Phi^k\Phi^l\partial_\mu\Phi^i\partial^\mu\Phi^j + \cdots\right) \tag{10.6}$$

giving

$$\tilde{\mathcal{A}}^{(4)}_{i_1 i_2 i_3 i_4}(s,t,u) = tR_{i_1 i_4 i_3 i_2} + uR_{i_1 i_3 i_4 i_2} + O(s^2, t^2, st)\,. \tag{10.7}$$

We can compare with the string theory amplitude (10.3) if we drop the first three terms, which in the low energy limit come from tachyon and graviton exchange. We then formally obtain our curvature formula (1.1). We say 'formally', because we have not addressed the issues of regularization. Assuming that those issues can be handled, our proof of the curvature formula becomes a point of support for the effective action assumption.

11 Discussion

We conclude with brief remarks on two topics: the possibilty of a general bound on the sectional curvature and the extension of the curvature formula to neighborhoods of CFT's with continuous symmetries.

Formulas (1.1), (1.3) express the curvature of the space of CFTs in terms of intrinsic CFT quantities — the four-point correlation functions. The correlation functions of a CFT satisfy reflection positivity, conformal invariance, and crossing symmetry. One might hope to use these properties to say something about the geometry of the space of CFTs.

One possibility is that reflection positivity of the four-point functions implies a bound on the sectional curvatures. Let directions $i = 1, 2$ be mutually orthogonal. From the curvature formula (1.1) we can write the sectional curvature in the 1-2 plane as

$$R_{2112} = \lim_{\epsilon \to 0} \int_{|z-1|>\epsilon,\, \epsilon<|z|<\epsilon^{-1}} \frac{d^2 z}{2\pi} \ln|1-z| \langle \phi_2(\infty)\phi_1(z)\phi_1(1)\phi_2(0)\rangle_c \tag{11.1}$$

where

$$\langle \phi_2(\infty)\phi_1(z)\phi_1(1)\phi_2(0)\rangle_c = \langle \phi_2(\infty)\phi_1(z)\phi_1(1)\phi_2(0)\rangle - |1-z|^{-4}\,. \tag{11.2}$$

For simplicity we have assumed no relevant operators. We have chosen this form of the curvature formula in order that the four-point function have the form appropriate for reflection positivity under the reflection $z \to 1/\bar{z}$ of the radial quantization. The full four-point function satisfies reflection positivity, but the connected four-point function does not, because of the subtraction. We have not managed to find a way around this obstacle. The logarithm in the integrand is another potential difficulty, but one might hope to get around it by using global conformal transformations.

Next, we discuss the cases that our curvature formula does not cover — the neighborhoods of CFT's with continuous symmetries. To handle these cases, one would have to relax our assumptions 1 and 3. At the symmetry point, one would have to allow for the conserved currents to occur in the OPE's of the perturbations ϕ_i. As discussed in appendix A, this would imply that some linear combinations of the ϕ_i become total derivatives away from the symmetry point. Therefore to cover the neighborhood of the symmetry point, one must allow from the start for perturbations that are total derivatives. To derive the curvature formula at the symmetry point, one would have to deal with the logarithmic divergence in the integral over η due to the occurrence of the current in the intermediate channels. One would also have to deal with the effects of the current on the conformal transformation properties of the regularization. To allow for total derivative perturbations ϕ_i, one will face further technical complications stemming from their conformal dimensions being different from two.

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A Bulk and boundary marginally redundant operators

In this appendix we elaborate on the meaning of assumptions 3 and 3b. We show that, for a CFT with continuous symmetry and charged perturbations ϕ_i, some linear combinations of the ϕ_i become redundant, at first order in the perturbation. We also show that the trace anomaly will contain a current term. We give analogous results for the boundary case.

We consider a reference CFT that satisfies assumptions 1 and 2 but not the assumption 3. Assumption 2 in particular excludes the dimension 2 current-current primaries $: J_m \bar{J}_n :$ from the $\phi_i \phi_j$ OPE. This implies that only holomorphic or only antiholomorphic currents appear in this OPE. The situation excluded by assumption 3 is therefore a reference CFT with perturbations charged under a chiral symmetry group.[6] Without loss of generality we restrict ourselves to the situation when the $\phi_i \phi_j$ OPE includes the holomorphic currents $J_m(z)$ and no relevant operators,

$$\phi_i(z)\,\phi_j(0) \sim \frac{1}{z\bar{z}^2} C_{ij}^m J_m(z)\,. \tag{A.1}$$

Assuming a real basis in the space of currents we normalize them as

$$\langle J_m(z) J_n(w)\rangle = -\frac{\delta_{mn}}{(z-w)^2} \tag{A.2}$$

so that the OPE coefficients C_{ij}^m are real and satisfy

$$C_{ij}^m = -C_{mij}\,, \qquad C_{ij}^m = -C_{ji}^m\,. \tag{A.3}$$

The three-point functions are

$$\langle\, \phi_i(z_1)\,\phi_j(z_2)\,J_m(z)\,\rangle_c = C_{mij}|z_1 - z_2|^{-4}(z_1 - z_2)(z_1 - z)^{-2}(z_2 - z)^{-2}\,. \tag{A.4}$$

In the reference CFT,

$$\langle \bar{T}(\bar{w}) J_m(z', \bar{z}')\phi_i(z, \bar{z})\rangle = 0\,. \tag{A.5}$$

Now we perturb by $\lambda^j \phi_j$. At first order, this three-point function becomes

$$\langle \bar{T}(\bar{w}) J_m(z', \bar{z}')\phi_i(z, \bar{z})\rangle_1 = \int \frac{d^2\xi}{2\pi} \langle \lambda^j \phi_j(\xi, \bar{\xi})\bar{T}(\bar{w}) J_m(z', \bar{z}')\phi_i(z, \bar{z})\rangle$$

[6] We thank D. Kutasov for a comment clarifying the point that the marginal couplings must develop a non-zero beta function if both holomorphic and anti-holomorphic currents are present in their OPE's. In [1], it was claimed that conformal invariance is broken away from the symmetry point when dimension one currents are present in the OPE of the perturbing fields. Presumably, it was implicitly assumed that both chiral and antichiral conserved currents occur in the OPE.

$$
\begin{aligned}
&= \frac{C_{mij}\lambda^j}{(z'-z)(\bar w-\bar z)^2}\int \frac{d^2\xi}{2\pi}\frac{1}{(z'-\xi)(z-\xi)(\bar w-\bar\xi)^2}\\
&= \frac{1}{2}C_{mij}\lambda^j\frac{1}{(z'-z)^2(\bar w-\bar z)^2}\left[\frac{1}{\bar w-\bar z}-\frac{1}{\bar w-\bar z'}\right]. \qquad \text{(A.6)}
\end{aligned}
$$

We see that $\bar T$ remains anti-holomorphic, so local conformal invariance is unbroken. Moreover, the correlation function decays as $\bar w^{-4}$ so global convormal invariance also remains unbroken. From the $(\bar w-\bar z')^{-1}$ singularity we obtain

$$
\partial_{\bar z}J_m(z,\bar z) = -\frac{1}{2}C_{mij}\lambda^j\phi_i(z,\bar z) \qquad \text{(A.7)}
$$

which means that, for every symmetry broken by the perturbation, there is a redundant field, given by the right hand side. Using (A.6) and the Ward identity for the stress-energy tensor we find a term in the trace anomaly

$$
\Theta(z,\bar z)\sim \lambda^j C_{mij}\partial_{\bar z}\lambda^i J_m(z,\bar z)\,. \qquad \text{(A.8)}
$$

Comparing to the general expression (4.2) for the trace anomaly, we see that the coefficients $C_i^m(\lambda)$ satisfy $\partial_i C_j^m(0)=C_{ij}^m$. We remark that the anomalous dimensions of the redundant operators come from this term in the trace anomaly, not from the beta function, which is zero. Explicitly, the scaling dimension matrix for the J_m is

$$
\Delta_n^m = \delta_n^m - \frac{1}{4}C_{nij}\lambda^j C^{mi}{}_k\lambda^k \qquad \text{(A.9)}
$$

through the second order in the couplings.

A simple example is given by the $c=1$ gaussian model at the self-dual point, which is the SU(2) WZW model with $k=1$. This example and more general toroidal examples were discussed in [13] (see section 9 in particular). We take as perturbations the fields $\phi_i = J_i(z)\bar J_3(\bar z)$, $i=1,2,3$. The symmetry currents are $J_i(z)$. The field ϕ_3 is the perturbation which changes the radius of the free boson in the gaussian model. Any perturbation $\lambda^i\phi_i$ can be rotated by the SU(2) symmetry to to a perturbation by ϕ_3 only, so all the perturbations $\lambda^i\phi_i$ preserve conformal invariance and are equivalent to a gaussian model away from the self-dual point.

For concreteness, consider a perturbation by ϕ_3. Let $X_L(z)$ and $X_R(\bar z)$ be the chiral parts of the free boson field normalized as

$$
\langle X_L(z)X_L(w)\rangle = -\ln(z-w)\,,\qquad \langle X_R(\bar z)X_R(\bar w)\rangle = -\ln(\bar z-\bar w)\,. \qquad \text{(A.10)}
$$

The current $J_3(z)$ is

$$
J_3(z) = -\partial X_L(z)\,. \qquad \text{(A.11)}
$$

The spin 1 fields J_1 and J_2 are given in terms of exponentials of the free boson

$$
\begin{aligned}
J_1(z,\bar z) &= i\sqrt{2}:\cos(Q_L X_L(z)+Q_R X_R(\bar z)):\,,\\
J_2(z,\bar z) &= -i\sqrt{2}:\sin(Q_L X_L(z)+Q_R X_R(\bar z)): \qquad \text{(A.12)}
\end{aligned}
$$

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where

$$Q_L = \frac{1}{\sqrt{2}}\left(R + \frac{1}{R}\right), \quad Q_R = \frac{1}{\sqrt{2}}\left(R - \frac{1}{R}\right) \tag{A.13}$$

with $R = 1$ corresponding to the self dual radius. We further identify

$$\phi_i(z,\bar{z}) = -:J_i\bar{\partial}X_R:(z,\bar{z})\,. \tag{A.14}$$

The OPE coefficients C_{mij} at the self dual point are

$$C_{mij} = \sqrt{2}\epsilon_{mij}\,. \tag{A.15}$$

With the perturbation away from the self-dual radius, the fields J_1, J_2 stop being holomorphic. Their divergences become proportional to the fields ϕ_1 and ϕ_2 so that the latter are now redundant. Explicitly we have

$$\bar{\partial}J_1 = -i\sqrt{2}Q_R :\sin(Q_L X_L(z) + Q_R X_R(\bar{z})): \bar{\partial}X_R = -Q_R\phi_2 \tag{A.16}$$

$$\bar{\partial}J_2 = -i\sqrt{2}Q_R :\cos(Q_L X_L(z) + Q_R X_R(\bar{z})): \bar{\partial}X_R = Q_R\phi_1 \tag{A.17}$$

which matches with formula (A.7) upon identifying $\lambda_3 = R - R^{-1}$. The conformal dimensions of fields J_1, J_2 become

$$\Delta = \frac{1}{4}\left(R + \frac{1}{R}\right)^2, \bar{\Delta} = \frac{1}{4}\left(R - \frac{1}{R}\right)^2 \tag{A.18}$$

which agrees with the general formula (A.9).

For a general perturbation we have a family of CFT's parametrized by the λ^i. The group SU(2) acts on the acts on this family. The point $\lambda^i = 0$ is a fixed point of the action, so the group is a symmetry group of that CFT. Away from the fixed point, only a U(1) subgroup leaves the CFT fixed. The full SU(2) group generates an SU(2)/U(1) equivalence class. The redundant fields are the perturbations within the equivalence class. The situation for a general symmetry G is the same.

There are analogous phenomena in boundary CFT's. Let $\psi_a(x)$ be dimension 1 boundary fields and $\chi_m(x)$ be dimension 0 boundary fields. Suppose the dimension 0 fields appear in the OPE's of the dimension 1 fields,

$$\psi_a(x)\psi_b(0) \sim \frac{1}{x^2}C_{ab}^m\chi_m \tag{A.19}$$

We normalize the fields so that

$$\langle\psi_a(x)\psi_b(0)\rangle = \frac{1}{x^2}\delta_{ab}\,, \qquad \langle\chi_m(x)\chi_n(0)\rangle = \delta_{mn}\,. \tag{A.20}$$

The above OPE's imply the commutation relations

$$[\chi_m, \psi_a(x)] = (C_{ma}{}^b - C_m{}^b{}_a)\psi_b(x)\,. \tag{A.21}$$

The χ_m are Chan-Paton charge operators. The matrices $C_{ma}{}^b - C_m{}^b{}_a$ give the charges of the ψ_a. To first order in the perturbation $\lambda^b\psi_b$,

$$\begin{aligned}\langle T(z)\psi_a(x')\chi_m(y)\rangle_1 &= \lambda^b \int dx\,\langle T(z)\psi_b(x)\psi_a(x')\chi_m(y)\rangle \\ &= \lambda^b(C_{mb}{}^a - C_m{}^a{}_b)\left[\frac{1}{(z-x')^3} + \frac{1}{(z-x')^2(z-y)}\right].\end{aligned} \tag{A.22}$$

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This gives the OPE in the perturbed theory

$$T(z)\chi_m(0) \sim \frac{1}{z}\lambda^b({C_{mb}}^a - {C_m}^a{}_b)\psi_a(0) + \dots \tag{A.23}$$

which implies

$$\partial_x\chi_m(x) = \lambda^b({C_{mb}}^a - {C_m}^a{}_b)\psi_a(x)\,. \tag{A.24}$$

Again, for every broken Chan-Paton symmetry we have a redundant field given by the right hand side.

Next we derive the boundary trace anomaly $\theta(x)$, which satisfies the conservation equation

$$T(x) - \bar{T}(x) = \partial_x\theta(x)\,. \tag{A.25}$$

From (A.22) we calculate

$$\begin{aligned}\langle[T(x+i\epsilon) - \bar{T}(x-i\epsilon)]\psi_a(x')\chi_m(y)\rangle &= -2\pi i\lambda^b({C_{mb}}^a - {C_m}^a{}_b) \\ &\times\left[\frac{1}{2}\partial_x^2\delta(x-x') + \partial_{x'}\left(\frac{1}{x'-y}\delta(x-x')\right) + \frac{1}{(x'-y)^2}\delta(x-y)\right].\end{aligned} \tag{A.26}$$

We can read off the boundary trace anomaly from the highest derivative term on the right hand side,

$$\theta = -\pi\lambda^b\partial_x\lambda^a(C^m_{ba} - C^m_{ab})\chi_m\,. \tag{A.27}$$

As in the bulk, this term in the trace anomaly gives the anomalous dimensions of the redundant fields.

We should note that what we have described is the two dimensional physics underlying the so-called string Higgs effect (see e.g. [12] for a review). When the CFT is a string theory compactification, the spin 1 dimension 1 fields J_m give massless gauge fields in space-time. The charged dimension 2 scalar fields ϕ_i give massless scalars in space-time. These are the Higgs fields. A perturbation $\lambda^i\phi_i$ corresponds to giving a vacuum expectation value to the Higgs fields. The spin 1 fields acquire anomalous dimensions, which correspond to the masses of the W bosons. The redundant fields are the pure gauge directions that are eaten up by the W bosons. The boundary case is parallel.

B Details of the boundary curvature computation

B.1 Proof of identity (8.19)

We need to prove

$$\tilde{R}^\epsilon_{abcd} = -\tilde{R}^\epsilon_{abdc} \tag{B.1}$$

where

$$\tilde{R}^\epsilon_{abcd} = \int d\chi\; F_\epsilon(\chi)\,[\langle\,\psi_b(0)\psi_d(\chi)\psi_a(1)\psi_c(\infty)\,\rangle_c + \langle\,\psi_b(1)\psi_d(\chi)\psi_a(0)\psi_c(\infty)\,\rangle_c] \tag{B.2}$$

and $F_\epsilon(\chi)$ is defined by (8.17).

Making conformal transformations

$$x' = \begin{cases} \frac{(1-\chi)x}{x-\chi} & \chi(1-\chi) < 0 \\ 1 - \frac{(1-\chi)x}{x-\chi} & \chi(1-\chi) > 0\,, \end{cases} \tag{B.3}$$

we obtain

$$\langle\, \psi_b(0)\psi_d(\chi)\psi_a(1)\psi_c(\infty)\,\rangle_c = \begin{cases} -\langle\, \psi_b(0)\psi_d(\infty)\psi_a(1)\psi_c(1-\chi)\,\rangle_c & \chi(1-\chi) < 0 \\ \langle\, \psi_b(1)\psi_d(\infty)\psi_a(0)\psi_c(\chi)\,\rangle_c & \chi(1-\chi) > 0 \end{cases} \tag{B.4}$$

$$\langle\, \psi_b(1)\psi_d(\chi)\psi_a(0)\psi_c(\infty)\,\rangle_c = \begin{cases} -\langle\, \psi_b(1)\psi_d(\infty)\psi_a(0)\psi_c(1-\chi)\,\rangle_c & \chi(1-\chi) < 0 \\ \langle\, \psi_b(0)\psi_d(\infty)\psi_a(1)\psi_c(\chi)\,\rangle_c & \chi(1-\chi) > 0 \end{cases} \tag{B.5}$$

We calculate

$$\begin{aligned} \tilde{R}^{\epsilon}_{abdc} = & \int\limits_{\chi(1-\chi)<0} d\chi\; F_\epsilon(1-\chi)\;\; [\langle\, \psi_b(0)\psi_d(\infty)\psi_a(1)\psi_c(\chi)\,\rangle_c - \langle\, \psi_b(1)\psi_d(\infty)\psi_a(0)\psi_c(\chi)\,\rangle_c] \\ & \quad + \int\limits_{\chi(1-\chi)>0} d\chi\; F_\epsilon(\chi)\, [\langle\, \psi_b(1)\psi_d(\infty)\psi_a(0)\psi_c(\chi)\,\rangle_c - \langle\, \psi_b(0)\psi_d(\infty)\psi_a(1)\psi_c(\chi)\,\rangle_c] \\ & = -\int d\chi\; F_\epsilon(\chi)\, [\langle\, \psi_b(0)\psi_d(\infty)\psi_a(1)\psi_c(\chi)\,\rangle_c - \langle\, \psi_b(1)\psi_d(\infty)\psi_a(0)\psi_c(\chi)\,\rangle_c] \\ & = -\tilde{R}^{\epsilon}_{abcd} \end{aligned} \tag{B.6}$$

which gives (8.19).

B.2 The functions $F_\epsilon(\chi)$ and $\Delta F_\epsilon(\chi)$

The function $F_\epsilon(\chi)$, defined by (8.17), is given in terms of radial integrals over the region $\mathcal{R}^{\epsilon}_{+}$ at fixed slope $\chi = x_2/x_1$. The region $\mathcal{R}^{\epsilon}_{+}$ is defined by (8.18) and is depicted in figure 1. The coordinates are (x_1, x_2). The four squares are the regions $\epsilon < |x_{1,2}| < \epsilon^{-1}$. The upper curve is $x_2 = y_+(x_1)$ and the lower curve is $x_2 = y_-(x_1)$ where

$$y_+(x) = \frac{x+\epsilon}{1-\epsilon x}\,, \qquad y_-(x) = \frac{x-\epsilon}{1+\epsilon x}\,, \qquad y_-(x) = -y_+(-x) = y_+(x^{-1})^{-1}\,. \tag{B.7}$$

The region $\mathcal{R}^{\epsilon}_{+}$ consists of the interiors of the 2 squares on the right, minus the portion lying between the curves. The dotted rays mark the transitions where the radial integral over $\mathcal{R}^{\epsilon}_{+}$ is not a smooth function of the slope χ. The slopes of the dotted rays are labelled χ^{+}_{k}. The piece-wise continuous function $F_\epsilon(\chi)$ is given in table 1. It is non-smooth at $\chi = \chi^{+}_{k}$ and at $\chi = 1 - \chi^{+}_{k}$. The function $\Delta F_\epsilon(\eta)$ is defined in (8.25).

In table 1, the function $A(s)$ is

$$A(s) = -2\ln\left(\frac{1}{2} + \sqrt{\frac{1}{4} - s}\right) = 2s + s^2 + O(s^3)\,. \tag{B.8}$$

The left column of the table lists the values of χ where $F_\epsilon(\chi)$ is non-smooth, arranged in decreasing order from $\chi = +\infty$ to $\chi = -\infty$. The rows between two adjacent thresholds give the values of $F_\epsilon(\chi)$ and $\Delta F_\epsilon(\chi)$ for χ in the corresponding interval.

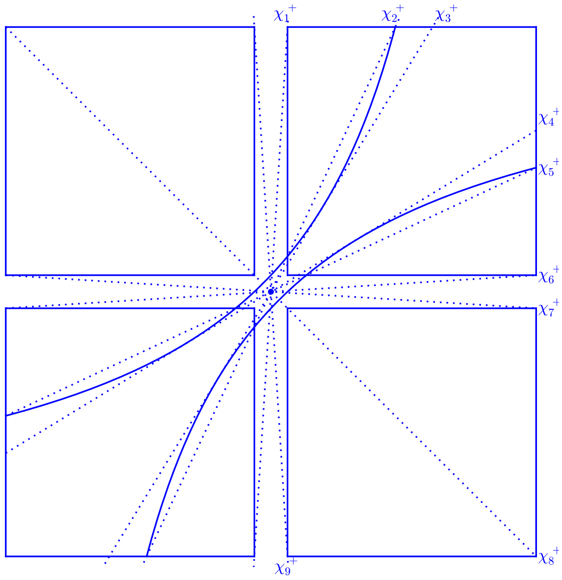

Figure 1. The region $\mathcal{R}_+^\epsilon$.

B.3 Computation of E_{abcd}

The error term E_{abcd} is defined by (8.24) as an integral over χ of four-point functions weighted by $\Delta F_\epsilon(\chi)$, which is given in table 1. We need to derive equation (8.36) which gives the asymptotic behavior of E_{abcd} in the limit $\epsilon \to 0$.

First we note the reflection symmetry

$$\Delta F_\epsilon(1-\chi) = -\Delta F_\epsilon(\chi)\,. \tag{B.9}$$

Next we note that the χ-intervals where ΔF_ϵ is identically zero of course make no contribution. By inspection, we see that the χ-intervals where ΔF_ϵ is explicitly written as $\mathcal{O}(\epsilon^2)$ in the table can also be neglected, because the four point functions are bounded there. We are left with four χ-intervals lying in the region $-1 \leq \chi \leq (1-\epsilon^2)/2$ and four χ-intervals lying in the reflected region. Now we can use the reflection symmetry to write

$$E_{abcd} = \int_{-1}^{(1-\epsilon^2)/2} d\chi \Delta F_\epsilon(\chi)[G_{abcd}(\chi) - G_{abcd}(1-\chi)] \tag{B.10}$$

where

$$G_{abcd}(\chi) = \langle \psi_b(0)\psi_c(\chi)\psi_a(1)\psi_d(\infty)\rangle_c - \langle \psi_a(0)\psi_c(\chi)\psi_b(1)\psi_d(\infty)\rangle_c\,. \tag{B.11}$$

In this region, $\Delta F_\epsilon \to 0$ except for a shrinking neighborhood of $\chi = 0$ where it diverges only logarithmically. Therefore non-negligible contributions to E_{abcd} come only from the singularities ni the four-point functions associated with relevant operators runing in the intermediate channels. Thus, up to terms vanishing in the limit $\epsilon \to 0$ we have

$$E_{abcd} = \sum_{a'} [G_{a'}^+ E^+(\Delta_{a'}) + G_{a'}^- E^-(\Delta_{a'})] \tag{B.12}$$

χ	$F_\epsilon(\chi)$	$\Delta F_\epsilon(\chi)$
	0	0
$1-\chi_9^+ = \epsilon^{-2}+1$		
	$\ln(\epsilon^{-2}) - \ln\lvert 1-\chi\rvert$	$-\ln(\epsilon^2\chi) = O(\epsilon^2)$
$\chi_1^+ = \epsilon^{-2}$		
	$\ln\lvert\chi\rvert - \ln\lvert 1-\chi\rvert$	0
$\chi_2^+ = \frac{2}{1-\epsilon^2}$		
	$\ln\lvert\chi\rvert - 3\ln\lvert 1-\chi\rvert + A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big)$	$-2\ln\lvert 1-\chi\rvert + A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big) = O(\epsilon^2)$
$1-\chi_8^+ = 2$		
	$\ln\lvert\chi\rvert - \ln\lvert 1-\chi\rvert + A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big)$	$A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big)$
$\chi_3^+ = \big(\sqrt{1+\epsilon^2}+\epsilon\big)^2$		
	$\ln(\epsilon^{-2}) + \ln\lvert 1-\chi\rvert$	$-\ln(\epsilon^2\chi) + \ln(\chi-1)^2$
$1-\chi_7^+ = 1+\epsilon^2$		
	0	0
$1-\chi_6^+ = 1-\epsilon^2$		
	$\ln(\epsilon^{-2}) + \ln\lvert 1-\chi\rvert$	$-\ln(\epsilon^2\chi) + \ln(\chi-1)^2$
$\chi_4^+ = \big(\sqrt{1+\epsilon^2}-\epsilon\big)^2$		
	$\ln\lvert\chi\rvert - \ln\lvert 1-\chi\rvert + A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big)$	$A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big)$
$1-\chi_5^+ = (1+\epsilon^2)/2$		
	$3\ln\lvert\chi\rvert - 3\ln\lvert 1-\chi\rvert$ $+A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big) - A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big)$	$2\ln\big(\frac{\chi}{1-\chi}\big) + A\big(\frac{\epsilon^2\chi}{(\chi-1)^2}\big) - A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big)$ $= O(\epsilon^2)$
$\chi_5^+ = (\chi_2^+)^{-1} = (1-\epsilon^2)/2$		
	$\ln\lvert\chi\rvert - \ln\lvert 1-\chi\rvert - A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big)$	$-A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big)$
$1-\chi_4^+ = 2\epsilon\sqrt{1+\epsilon^2} - 2\epsilon^2$		
	$\ln(\epsilon^2) - \ln\lvert\chi\rvert$	$\ln(\epsilon^2(1-\chi)) - \ln\chi^2$
$\chi_6^+ = (\chi_1^+)^{-1} = \epsilon^2$		
	0	0
$\chi_7^+ = -\epsilon^2$		
	$\ln(\epsilon^2) - \ln\chi^2$	$\ln(\epsilon^2(1-\chi)) - \ln\lvert\chi\rvert$
$1-\chi_3^+ = -2\epsilon\sqrt{1+\epsilon^2} - 2\epsilon^2$		
	$\ln\lvert\chi\rvert - \ln\lvert 1-\chi\rvert - A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big)$	$-A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big)$
$\chi_8^+ = -1$		
	$3\ln\lvert\chi\rvert - \ln\lvert 1-\chi\rvert - A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big)$	$2\ln\lvert\chi\rvert - A\big(\frac{\epsilon^2(1-\chi)}{\chi^2}\big) = O(\epsilon^2)$
$1-\chi_2^+ = (1+\epsilon^2)/(\epsilon^2-1)$		
	$\ln\lvert\chi\rvert - \ln\lvert 1-\chi\rvert$	0
$1-\chi_1^+ = -\epsilon^{-2}+1$		
	$\ln\lvert\chi\rvert + \ln(\epsilon^2)$	$\ln\lvert 1-\chi\rvert + \ln(\epsilon^2) = O(\epsilon^2)$
$\chi_9^+ = (\chi_7^+)^{-1} = -\epsilon^{-2}$		
	0	0

Table 1. The functions $F_\epsilon(\chi)$ and $\Delta F_\epsilon(\chi)$.

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where

$$E^{+}(\Delta) = \int_{\epsilon^2}^{\frac{1-\epsilon^2}{2}} d\chi\ \Delta F_\epsilon(\chi)\chi^{\Delta-2}\,, \tag{B.13}$$

$$E^{-}(\Delta) = \int_{-1}^{-\epsilon^2} d\chi\ \Delta F_\epsilon(\chi)(-\chi)^{\Delta-2} = \int_{\epsilon^2}^{1} d\chi\ \Delta F_\epsilon(-\chi)\chi^{\Delta-2}\,, \tag{B.14}$$

$$G^{+}_{a'} = [C^{a'}_{cb}C_{daa'} - C^{a'}_{ac}C_{bda'}](1 - a \leftrightarrow b)\,, \tag{B.15}$$

$$G^{-}_{a'} = [C^{a'}_{bc}C_{daa'} - C^{a'}_{ca}C_{bda'}](1 - a \leftrightarrow b)\,. \tag{B.16}$$

Changing the integration variable to $\chi = \epsilon u$ we calculate

$$E^{+}(\Delta) = \epsilon^{\Delta-1}\int_{\epsilon}^{\frac{1-\epsilon^2}{2\epsilon}} du\ k_{+}(u)u^{\Delta-2} = \epsilon^{\Delta-1}\int_{\epsilon}^{\frac{1-\epsilon^2}{2\epsilon}} du\ k_{+}(u)\partial_u\frac{u^{\Delta-1}}{\Delta-1} \tag{B.17}$$

$$E^{-}(\Delta) = \epsilon^{\Delta-1}\int_{\epsilon}^{\frac{1}{\epsilon}} du\ k_{-}(u)u^{\Delta-2} = \epsilon^{\Delta-1}\int_{\epsilon}^{\frac{1}{\epsilon}} du\ k_{-}(u)\partial_u\frac{u^{\Delta-1}}{\Delta-1} \tag{B.18}$$

where

$$k_{+}(u) = \Delta F_\epsilon(\epsilon u) = \begin{cases} \ln\left(\frac{1-\epsilon u}{u^2}\right) & \epsilon \le u \le 2\sqrt{1+\epsilon^2} - 2\epsilon \\ -A(\frac{1-\epsilon u}{u^2}) & 2\sqrt{1+\epsilon^2} - 2\epsilon \le u \le \frac{1-\epsilon^2}{2\epsilon} \end{cases} \tag{B.19}$$

$$k_{-}(u) = \Delta F_\epsilon(-\epsilon u) = \begin{cases} \ln\left(\frac{1+\epsilon u}{u^2}\right) & \epsilon \le u \le 2\sqrt{1+\epsilon^2} + 2\epsilon \\ -A(\frac{1+\epsilon u}{u^2}) & 2\sqrt{1+\epsilon^2} + 2\epsilon \le u \le \frac{1}{\epsilon}\,. \end{cases} \tag{B.20}$$

Integrating by parts, using the fact that the $k_{\pm}(u)$ are continuous within the range of integration, and dropping the contributions from the upper boundaries because they are $O(\epsilon^2)$, we get

$$E^{+}(\Delta) = \epsilon^{\Delta-1}\left[k_{+}(\epsilon)\frac{\epsilon^{\Delta-1}}{1-\Delta} + \int_{\epsilon}^{\frac{1-\epsilon^2}{2\epsilon}} du\ k'_{+}(u)\frac{u^{\Delta-1}}{1-\Delta}\right] \tag{B.21}$$

$$E^{-}(\Delta) = \epsilon^{\Delta-1}\left[k_{-}(\epsilon)\frac{\epsilon^{\Delta-1}}{1-\Delta} + \int_{\epsilon}^{\frac{1}{\epsilon}} du\ k'_{-}(u)\frac{u^{\Delta-1}}{1-\Delta}\right]\,. \tag{B.22}$$

Since

$$k_{\pm}(\epsilon) = \ln(\epsilon^{-2}) + O(\epsilon^2)\,, \tag{B.23}$$

we can write

$$E^{\pm}(\Delta) = \ln(\epsilon^{-2})\frac{(\epsilon^2)^{\Delta-1}}{1-\Delta} + O(\epsilon^{2\Delta}) + I^{\pm}(\Delta) \tag{B.24}$$

with

$$I^{+}(\Delta) = \epsilon^{\Delta-1}\int_{\epsilon}^{\frac{1-\epsilon^2}{2\epsilon}} du\ k'_{+}(u)\frac{u^{\Delta-1}}{1-\Delta} \tag{B.25}$$

$$I^{-}(\Delta) = \epsilon^{\Delta-1}\int_{\epsilon}^{\frac{1}{\epsilon}} du\ k'_{-}(u)\frac{u^{\Delta-1}}{1-\Delta}\,. \tag{B.26}$$

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Analyzing the behaviour of these integrals in the limit $\epsilon \to 0$ we find, up to terms vanishing in the limit $\epsilon \to 0$,

$$E^+(\Delta) = E^-(\Delta) = \ln(\epsilon^{-2})\frac{(\epsilon^2)^{\Delta-1}}{1-\Delta} - 2\frac{(\epsilon^2)^{\Delta-1}}{(1-\Delta)^2} + \frac{\epsilon^{\Delta-1}}{1-\Delta}\int_2^\infty du\ \left[-\partial_u A(u^{-2}) + 2u^{-1}\right] u^{\Delta-1}\,. \tag{B.27}$$

Therefore

$$E_{abcd} = \sum_{a'} E^+(\Delta_{a'})\left(G^+_{a'} + G^-_{a'}\right) = \sum_{a'} E^+(\Delta_{a'})\left(C^{a'}_{(ad)}C_{(bc)a'} - C^{a'}_{(ac)}C_{(bd)a'}\right) \tag{B.28}$$

which is equation (8.36).

C Distributional correlators of currents

In this appendix we construct the distributional three-point and four-point correlation functions of currents for the D0 brane example discussed in section 9. By translation invariance, the three-point function is a distribution in two real variables and the four-point function is a distribution in three real variables. Along the way we derive some useful identities on distributions.

C.1 Distributions in two variables and the three-point function

We define a distribution $\mathrm{PV}\frac{1}{xy}$ in the two real variables x and y by its action on test functions $f(x,y)$,

$$\left(f, \mathrm{PV}\frac{1}{xy}\right) = \lim_{\epsilon\to 0} \iint\limits_{|x|,|y|\geq\epsilon} \frac{f(x,y)}{xy} \tag{C.1}$$

which is equivalent to

$$\left(f, \mathrm{PV}\frac{1}{xy}\right) = \int\limits_0^\infty dx \int\limits_0^\infty dy\, \frac{1}{xy}(1-R_x)(1-R_y)f(x,y) \tag{C.2}$$

where

$$R_x : (x,y) \mapsto (-x,y) \tag{C.3}$$

$$R_y : (x,y) \mapsto (x,-y)\,. \tag{C.4}$$

Next define

$$\left(f, \mathrm{PV}\frac{1}{x(y-x)}\right) = \lim_{\epsilon\to 0} \iint\limits_{|x|,|y-x|\geq\epsilon} \frac{f(x,y)}{x(y-x)} \tag{C.5}$$

$$\left(f, \mathrm{PV}\frac{1}{y(x-y)}\right) = \lim_{\epsilon\to 0} \iint\limits_{|y|,|y-x|\geq\epsilon} \frac{f(x,y)}{y(x-y)} \tag{C.6}$$

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which are equivalent to

$$\left(f, \mathrm{PV}\frac{1}{x(y-x)}\right) = \int_0^\infty dx \int_0^\infty dy\, \frac{1}{xy}(1-R_x)(1-R_y)f(x, y+x)\,, \tag{C.7}$$

$$\left(f, \mathrm{PV}\frac{1}{y(x-y)}\right) = \int_0^\infty dx \int_0^\infty dy\, \frac{1}{xy}(1-R_x)(1-R_y)f(x+y, y)\,. \tag{C.8}$$

The following useful identities follow directly from the definitions

$$\mathrm{PV}\frac{1}{xy} - \mathrm{PV}\frac{1}{x(y-x)} - \mathrm{PV}\frac{1}{y(x-y)} = \pi^2\delta(x)\delta(y)\,, \tag{C.9}$$

$$\mathrm{PV}\frac{1}{xy} - \mathrm{PV}\frac{1}{x(y+x)} - \mathrm{PV}\frac{1}{y(x+y)} = -\pi^2\delta(x)\delta(y)\,. \tag{C.10}$$

We now turn to constructing the distributional three-point function of currents on the boundary. At finite separations the three-point function on the boundary is

$$\langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle = -ikf^{abc}\frac{1}{x_{12}x_{13}x_{32}}\,. \tag{C.11}$$

We want a distributional regularization

$$\left[\frac{1}{x_{12}x_{13}x_{32}}\right] \tag{C.12}$$

of the rational function, which must be fully antisymmetric in x_1, x_2, x_3 because the three-point function is symmetric. By translation invariance, we can think of such a distribution as a distribution in two variables x_2 and x_3, treating x_1 as a parameter.

Define

$$\left[\frac{1}{x_{12}x_{13}x_{32}}\right]_1 = \partial_2\mathrm{PV}\frac{1}{x_{32}x_{13}} + \partial_3\mathrm{PV}\frac{1}{x_{32}x_{12}}\,. \tag{C.13}$$

Using the identities (C.9), (C.10) we find that this distribution transforms in the following way under permutations σ_{12}, σ_{23}:

$$\sigma_{12}\left[\frac{1}{x_{12}x_{13}x_{32}}\right]_1 = -\left[\frac{1}{x_{12}x_{13}x_{32}}\right]_1 + \pi^2\partial_3(\delta(x_{12})\delta(x_{13})) \tag{C.14}$$

$$\sigma_{23}\left[\frac{1}{x_{12}x_{13}x_{32}}\right]_1 = -\left[\frac{1}{x_{12}x_{13}x_{32}}\right]_1\,. \tag{C.15}$$

It follows that the distribution

$$\left[\frac{1}{x_{12}x_{13}x_{32}}\right] = \partial_2\mathrm{PV}\frac{1}{x_{32}x_{13}} + \partial_3\mathrm{PV}\frac{1}{x_{32}x_{12}} + \frac{\pi^2}{3}(\partial_2-\partial_3)\delta(x_{12})\delta(x_{13}) \tag{C.16}$$

is fully antisymmetric. We thus set

$$\langle J^a(x_1)J^b(x_2)J^c(x_3)\rangle = -ikf^{abc}\left[\frac{1}{x_{12}x_{13}x_{32}}\right]. \tag{C.17}$$

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C.2 Distributions in three variables and the four-point function

We define the following distributions in three real variables x, y, z:

$$\left(f, \mathrm{PV}\frac{1}{xyz}\right) = \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \, \frac{1}{xyz}(1-R_x)(1-R_y)(1-R_z) f(x,y,z)\,, \tag{C.18}$$

$$\begin{aligned}\left(f, \mathrm{PV}\frac{1}{(x-x')(x-y)(y-z)}\right) &= \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \, \frac{1}{xyz} \\ &\quad \times (1-R_x)(1-R_y)(1-R_z) f(x'-x, x'-x-y, x'-x-y-z)\end{aligned} \tag{C.19}$$

where

$$R_z : (x,y,z) \mapsto (x,y,-z) \tag{C.20}$$

and x' is a parameter. Distributions with other factors in the denominator are defined analogously. Identities of the following type hold

$$\begin{aligned}\mathrm{PV}\frac{1}{(x-x')(x-y)(y-z)} &= \mathrm{PV}\frac{1}{(x-x')(x-y)(x-z)} \\ &\quad + \mathrm{PV}\frac{1}{(x-x')(x-z)(y-z)} - \pi^2 \delta(x-y)\delta(y-z)\mathrm{PV}\frac{1}{x'-x}\,.\end{aligned} \tag{C.21}$$

That is, the identities (C.9) and (C.10) can be used inside the PV symbol.

The connected four-point function of currents on the boundary is, at separated points,

$$\langle J^a(x_1) J^b(x_2) J^c(x_3) J^d(x_4)\rangle_c = \frac{k f^{ac}{}_s f^{bsd}}{x_{13}x_{12}x_{34}x_{24}} + \frac{k f^{ad}{}_s f^{bcs}}{x_{14}x_{12}x_{23}x_{43}}\,. \tag{C.22}$$

We need to extend the rational functions to distributions in three variables, which we take to be x_2, x_3, x_4, leaving x_1 as a parameter. The full distributional four-point function must be symmetric under simultaneous permutations of x_i and the corresponding group indices. By a slight abuse of terminology we will refer to this symmetry as crossing symmetry.

We start by defining

$$D_1 \equiv \left[\frac{1}{x_{12}x_{23}x_{34}x_{42}}\right]_1 = -\partial_4 \mathrm{PV}\frac{1}{x_{12}x_{34}x_{23}} - \partial_3 \mathrm{PV}\frac{1}{x_{12}x_{42}x_{43}}\,, \tag{C.23}$$

$$D_2 \equiv \left[\frac{1}{x_{13}x_{23}x_{34}x_{42}}\right]_1 = -\partial_4 \mathrm{PV}\frac{1}{x_{13}x_{24}x_{23}} - \partial_2 \mathrm{PV}\frac{1}{x_{13}x_{42}x_{34}}\,, \tag{C.24}$$

$$D_3 \equiv \left[\frac{1}{x_{14}x_{23}x_{34}x_{42}}\right]_1 = -\partial_3 \mathrm{PV}\frac{1}{x_{14}x_{23}x_{42}} - \partial_2 \mathrm{PV}\frac{1}{x_{14}x_{32}x_{34}}\,, \tag{C.25}$$

and then defining

$$\left[\frac{1}{x_{13}x_{12}x_{34}x_{24}}\right]_1 = D_2 - D_1\,, \tag{C.26}$$

$$\left[\frac{1}{x_{14}x_{12}x_{23}x_{34}}\right]_1 = D_3 - D_1\,, \tag{C.27}$$

$$\left[\frac{1}{x_{13}x_{14}x_{23}x_{24}}\right]_1 = D_3 - D_2\,. \tag{C.28}$$

The distributions $D_2 - D_1$ and $D_3 - D_1$ regularize the rational functions which appear in (C.22). We next turn to their behaviour under permutations. We find

$$\sigma_{12}(D_2 - D_1) = D_1 - D_3 + \Xi_1\,, \tag{C.29}$$

$$\sigma_{13}(D_2 - D_1) = D_3 - D_2 + \Xi_2\,, \tag{C.30}$$

$$\sigma_{14}(D_3 - D_1) = D_2 - D_3 + \Xi_3\,, \tag{C.31}$$

where Ξ_i are contact terms which can be computed using (C.9) and (C.10):

$$\begin{aligned}
\Xi_1 &= -\pi^2\partial_4\left(\delta(x_{14})\delta(x_{34})\mathrm{PV}\frac{1}{x_{23}} + \delta(x_{13})\delta(x_{23})\mathrm{PV}\frac{1}{x_{34}}\right)\\
&\quad + \pi^2\partial_3\left(\delta(x_{21})\delta(x_{41})\mathrm{PV}\frac{1}{x_{43}} - \delta(x_{23})\delta(x_{34})\mathrm{PV}\frac{1}{x_{41}}\right),\\
\Xi_2 &= -\pi^2\partial_4\left(\delta(x_{42})\delta(x_{41})\mathrm{PV}\frac{1}{x_{32}} + \delta(x_{31})\delta(x_{21})\mathrm{PV}\frac{1}{x_{24}}\right)\\
&\quad + \pi^2\partial_2\left(\delta(x_{31})\delta(x_{14})\mathrm{PV}\frac{1}{x_{42}} - \delta(x_{32})\delta(x_{42})\mathrm{PV}\frac{1}{x_{41}}\right),\\
\Xi_3 &= \pi^2\partial_3\left(\delta(x_{31})\delta(x_{23})\mathrm{PV}\frac{1}{x_{42}} + \delta(x_{41})\delta(x_{12})\mathrm{PV}\frac{1}{x_{23}}\right)\\
&\quad + \pi^2\partial_2\left(-\delta(x_{41})\delta(x_{31})\mathrm{PV}\frac{1}{x_{32}} + \delta(x_{42})\delta(x_{23})\mathrm{PV}\frac{1}{x_{31}}\right).
\end{aligned} \tag{C.32}$$

To satisfy the crossing symmetry we modify $D_2 - D_1$ according to the ansatz

$$\left[\frac{1}{x_{13}x_{12}x_{34}x_{24}}\right] = D_2 - D_1 + \xi \tag{C.33}$$

with

$$\begin{aligned}
\xi &= A_1\delta_{234}\left[\frac{1}{x_{12}^2}\right] + A_2\delta_{134}\left[\frac{1}{x_{12}^2}\right] + A_3\delta_{124}\left[\frac{1}{x_{13}^2}\right] + A_4\delta_{123}\left[\frac{1}{x_{41}^2}\right] + B_{123}\partial_3\left(\delta_{123}\mathrm{PV}\frac{1}{x_{43}}\right)\\
&\quad + B_{132}\partial_2\left(\delta_{123}\mathrm{PV}\frac{1}{x_{43}}\right) + B_{124}\partial_4\left(\delta_{124}\mathrm{PV}\frac{1}{x_{34}}\right) + B_{142}\partial_2\left(\delta_{124}\mathrm{PV}\frac{1}{x_{34}}\right) + B_{134}\partial_4\left(\delta_{134}\mathrm{PV}\frac{1}{x_{24}}\right)\\
&\quad + B_{143}\partial_3\left(\delta_{134}\mathrm{PV}\frac{1}{x_{34}}\right) + B_{234}\partial_4\left(\delta_{234}\mathrm{PV}\frac{1}{x_{14}}\right) + B_{243}\partial_3\left(\delta_{234}\mathrm{PV}\frac{1}{x_{13}}\right),
\end{aligned} \tag{C.34}$$

defining

$$\delta_{ijk} = \delta(x_{ij})\delta(x_{jk})\,, \qquad \left[\frac{1}{x_{ij}^2}\right] = \partial_j\mathrm{PV}\frac{1}{x_{ij}}\,. \tag{C.35}$$

Similarly we make the ansatz

$$\left[\frac{1}{x_{14}x_{12}x_{23}x_{34}}\right] = D_3 - D_1 - \eta \tag{C.36}$$

where η has the same form as ξ above with the coefficients A_i replaced by P_i and B_{ijk} replaced by Q_{ijk}. The crossing symmetry requirement implies the equations

$$\eta - \sigma_{12}\xi = \Xi_1\,, \tag{C.37}$$

$$\eta + \xi + \sigma_{13}\xi = -\Xi_2\,, \tag{C.38}$$

$$\xi + \eta + \sigma_{14}\eta = \Xi_3\,. \tag{C.39}$$

Solving these equations we obtain that the only nonvanishing coefficients present in ξ and η are

$$A_1 = A_2 = A_3 = -\frac{\pi^2}{3}\,, \qquad\qquad A_4 = \frac{2\pi^2}{3}\,, \tag{C.40}$$

$$P_1 = P_2 = P_4 = -\frac{\pi^2}{3}\,, \qquad\qquad P_3 = \frac{2\pi^2}{3}\,, \tag{C.41}$$

$$B_{234} = Q_{243} = \pi^2\,. \tag{C.42}$$

The full distributional four-point function is

$$\langle J^a(t_1)J^b(t_2)J^c(t_3)J^d(t_4)\rangle_c = k f^{ac}{}_s f^{bsd}\left[\frac{1}{x_{13}x_{12}x_{34}x_{24}}\right] + k f^{ad}{}_s f^{bcs}\left[\frac{1}{x_{14}x_{12}x_{23}x_{43}}\right] \tag{C.43}$$

where

$$\begin{aligned}\left[\frac{1}{x_{13}x_{12}x_{34}x_{24}}\right] &= \left[\frac{1}{x_{13}x_{12}x_{34}x_{24}}\right]_1 + \pi^2\partial_4(\delta_{234})\mathrm{PV}\frac{1}{x_{12}} \\ &\quad + \frac{\pi^2}{3}\left(2\delta_{123}\left[\frac{1}{x_{14}^2}\right] - \delta_{234}\left[\frac{1}{x_{12}^2}\right] - \delta_{134}\left[\frac{1}{x_{32}^2}\right] - \delta_{124}\left[\frac{1}{x_{32}^2}\right]\right)\,, \end{aligned}\tag{C.44}$$

$$\begin{aligned}\left[\frac{1}{x_{14}x_{12}x_{23}x_{43}}\right] &= \left[\frac{1}{x_{14}x_{12}x_{23}x_{43}}\right]_1 - \pi^2\partial_3\delta_{234}\mathrm{PV}\frac{1}{x_{12}} \\ &\quad + \frac{\pi^2}{3}\left(-2\delta_{124}\left[\frac{1}{x_{13}^2}\right] + \delta_{234}\left[\frac{1}{x_{12}^2}\right] + \delta_{134}\left[\frac{1}{x_{32}^2}\right] + \delta_{123}\left[\frac{1}{x_{41}^2}\right]\right)\,. \end{aligned}\tag{C.45}$$

References

[1] D. Kutasov, *Geometry on the space of conformal field theories and contact terms*, *Phys. Lett.* **B 220** (1989) 153 [INSPIRE].

[2] L.J. Dixon, V. Kaplunovsky and J. Louis, *On effective field theories describing* (2, 2) *vacua of the heterotic string*, *Nucl. Phys.* **B 329** (1990) 27 [INSPIRE].

[3] J. de Boer, J. Manschot, K. Papadodimas and E. Verlinde, *The chiral ring of* AdS_3/CFT_2 *and the attractor mechanism*, *JHEP* **03** (2009) 030 [arXiv:0809.0507] [INSPIRE].

[4] D. Friedan, *A tentative theory of large distance physics*, *JHEP* **10** (2003) 063 [hep-th/0204131] [INSPIRE].

[5] D. Friedan and A. Konechny, *Gradient formula for the* β*-function of* 2*d quantum field theory*, *J. Phys.* **A 43** (2010) 215401 [arXiv:0910.3109] [INSPIRE].

[6] E. Rabinovici, *Spontaneous breaking of space-time symmetries*, *Lect. Notes Phys.* **737** (2008) 573 [arXiv:0708.1952] [INSPIRE].

[7] H. Osborn, *Weyl consistency conditions and a local renormalization group equation for general renormalizable field theories*, *Nucl. Phys.* **B 363** (1991) 486 [INSPIRE].

[8] D. Kutasov, M. Mariño and G.W. Moore, *Some exact results on tachyon condensation in string field theory*, *JHEP* **10** (2000) 045 [hep-th/0009148] [INSPIRE].

[9] D. Friedan and A. Konechny, *On the boundary entropy of one-dimensional quantum systems at low temperature*, *Phys. Rev. Lett.* **93** (2004) 030402 [hep-th/0312197] [INSPIRE].

[10] I. Affleck and A.W. Ludwig, *Universal noninteger 'ground state degeneracy' in critical quantum systems*, Phys. Rev. Lett. **67** (1991) 161 [INSPIRE].

[11] A. Recknagel and V. Schomerus, *Boundary deformation theory and moduli spaces of D-branes*, Nucl. Phys. **B 545** (1999) 233 [hep-th/9811237] [INSPIRE].

[12] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, Phys. Rept. **244** (1994) 77 [hep-th/9401139] [INSPIRE].

[13] G.W. Moore, *Finite in all directions*, hep-th/9305139 [INSPIRE].